VIRUDHUNAGAR S.VELLAICHAMY NADAR POLYTECHNIC COLLEGE
(Affiliated to Directorate of Technical Education, Chennai 600 025)
Virudhunagar - 626 001

MISSION

"Our Mission is to Produce Disciplined and Quality Technocrats through Academic Programme of Noted Excellence to Serve the Society"

VISION

"To be an Institute of Excellence in Technical Education and Training of individuals , Focusing on the needs of the Nation and Society in tune with Technological Developments"

COURSE MATERIAL	
SCHEME	N1
PROGRAMME	BASIC SCIENCE
TERM	I
COURSE	ENGINEERING MATHEMATICS
PREPARED BY	P.Parameswaran, M.Sc., M.Phil., B.Ed B.Jaihind, M.Sc., M.Phil., PGDCA A.Kalaiselvi, M.Sc., M.Phil., K.Sriram, M.Sc., M.Phil.,
VERIFIED BY	

DEPARTMENT OF BASIC SCIENCE

VIRUDHUNAGAR S.VELLAICHAMY NADAR POLYTECHNIC COLLEGE

(AUTONOMOUS)

(Affiliated to Department of Technical Education, Chennai-25)

VIRUDHUNAGAR – 626 001

N1 - SCHEME

(Implements from the Academic year 2019-2020 onwards)

Programme : **All Programmes of Diploma in Engineering and Technology except DMOP.**

Course code : **N1BE102**

Term : **I Term**

Course Name : **ENGINEERING MATHEMATICS**

TEACHING AND SCHEME OF EXAMINATION

Number of weeks per Term: 15 weeks

Course	Instructions		Examination			
	Hours / Week	Hours / Term	Marks			Duration
			Internal Assessment	End Examination	Total	
ENGINEERING MATHEMATICS	7 Hrs.	105 Hrs.	25	75	100	3 Hrs.

Topics and Allocation of Hours:

UNIT	Topic	Time
1	Algebra and Basics of Complex Number	18 Hrs.
2	Complex Number and Trigonometry	18 Hrs.
3	Differential Calculus	20 Hrs.
4	Partial Differentiation & Integration	20 Hrs.
5	Vector Algebra	19 Hrs.
	REVISION, ASSESMENT TEST AND MODEL EXAM	10 Hrs.
	Total	**105 Hrs**

N1BE102 ENGINEERING MATHEMATICS

DETAILED SYLLABUS

Contents: Theory

UNIT	NAME OF TOPIC	Time
I	**ALGEBRA &COMPLEX NUMBER**	6 Hrs.
	1.1 DETERMINANTS: Definition and expansion of determinants of order 2 and 3.Properties of determinants (NOT for examination)-Solution of simultaneous equations using Cramer's rule (in 2 and 3 unknowns) - Simple problems.	
	1.2 MATRICES : Adjoint of a matrix and Inverse of a matrix of order 2 and 3 – simple problems related to Adjoint and Inverse only (Algebra of matrices and properties of Inverse problems NOT for examination) – Rank of a matrix by determinant form.(matrix of order 3 x 4) – simple problems –	6 Hrs.
	Definitions of Factorial notation, Definition of Permutation and Combination (NOT for examination). Binomial Theorem for rational index upto -2(statement only)- Expansion only for -1, and -2 (Approximation problems NOT for examination).	6 Hrs.
	1.3 ALGEBRA OF COMPLEX NUMBERS: Definition - Real and Imaginary parts, conjugates, Modulus and amplitude form, Polar form(definition only),multiplication and division of complex numbers (geometrical proof not needed) – Simple problems.	
II	**COMPLEX NUMBER & TRIGONOMETRY**	
	2.1 DE MOIVRE'S THEOREM: DeMoivre's Theorem (statement only) - Related simple problems.	6 Hrs.
	2.2 ROOTS OF A COMPLEX NUMBER: Find the nth roots of unity-solving equation of the form $x^n \pm 1 = 0$ where $n \leq 7$ - Simple Problems.	6 Hrs.
	2.3 COMPOUND ANGLES: Expansion of sin(A±B), cos(A±B) and tan(A±B) (without proof) - simple problems related to expansion only. Multiple angle of 2A and 3A (Formula only) (Problem NOT for examination) - Sum and Product formulae - Simple Problems.	6 Hrs.

III	**DIFFERENTIAL CALCULUS** **3.1 LIMITS**: Definition of limits. Problems using the following results: (i) $\underset{x\to a}{\text{Lt}} \frac{x^n-a^n}{x-a} = na^{n-1}$ (ii) $\lim_{\theta\to 0}\frac{sin\theta}{\theta}$ =1 (iii) $\lim_{\theta\to 0}\frac{tan\theta}{\theta}$ =1 (θ in radians) (results only) - Simple Problems. **3.2 DIFFERENTIATION:** Definition – Differentiation of x^n, sinx, cosx, tanx, cosecx, secx, cotx, logx, e^x, u±v, uv, uvw, u/v (v≠0) - Simple Problems using above functions and properties. **3.3 DIFFERENTIATION METHODS**: Differentiation of function of function(Chain Rule) - Successive Differentiation up to second order (Parametric form not included) - Simple Problems.	6 Hrs. 7 Hrs. 7 Hrs.
IV	**PARTIAL DIFFERENTIATION & INTEGRATION** **4.1 PRTIAL DIFFERENTIATION**: Definition –Partial differentiation of two variables up to second order only - Simple Problems. **4.2 INTEGRATION**: Introduction – Definition of integration- Integral values using reverse process of differentiation – Integration using decomposition method - Simple Problems. **4.3 INTEGRATION BY SUBSTITUTION:** Integrals of the form $\int[f(x)]$ f'(x)dx, n≠ -1, $\int\frac{f'(x)}{f(x)}$dx and $\int F[f(x)]^n f'(x)dx$ - Simple Problems.	7 Hrs. 6 Hrs. 7 Hrs.
V	VECTOR ALGEBRA **5.1 VECTOR INTRODUCTION:** Definition of Vector –types, addition and subtraction of vectors, Properties of addition and subtraction - Position vector. Resolution of vector in two and three dimensions. Direction cosines, Direction ratios - Simple problems. **5.2 SCALAR PRODUCT OF VECTORS**: Definition of Scalar product of two vectors, Geometrical meaning - properties – angle between two vectors - Simple Problems. **5.3 VECTOR PRODUCT OF VECTORS:** Definition of vector product of two vectors –Geometrical meaning –properties angle between two vectors – Unit vectors perpendicular to two vectors - Simple Problems.	6 Hrs. 7 Hrs. 6 Hrs.

Reference Book: Tamil Nadu state Hr.sec.school book for I year Volume I and II
Tamil Nadu state Hr.sec.school book for II year Volume I and II

UNIT I ALGEBRA AND COMPLEX NUMBER

1.1 DETERMINANTS

Definition:

Determinant is a square arrangement of numbers (real or complex) within two vertical lines.

Example: $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

Order:

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, consisting of two rows and two columns is called a determinant of second order

the value of the determinant is $\Delta = a_1 b_2 - b_1 a_2$

Example: $|A| = \begin{vmatrix} 2 & -5 \\ 1 & 3 \end{vmatrix}$

| A | = (2) (3) –(1) (– 5)

| A | = 6 + 5

i.e, Δ = 11

Determinant of third order.

The expression $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ consisting of three rows and three columns is called a determinant of third order

The value of the determinant is obtained by expanding the determinant through any row or column with proper sign attached starting from a_1 the first row first column elements, we will have positive and negative sign alternately

Example: Let A = $\begin{bmatrix} 1 & -1 & 3 \\ 0 & 4 & 2 \\ 11 & 5 & -3 \end{bmatrix}$

$$|A| = 1\begin{vmatrix} 4 & 2 \\ 5 & -3 \end{vmatrix} - (-1)\begin{vmatrix} 0 & 2 \\ 11 & -3 \end{vmatrix} + 3\begin{vmatrix} 0 & 4 \\ 11 & 5 \end{vmatrix}$$

= 1 (– 12 – 10) + 1 (0 – 22) + 3 (0 – 44)

= – 22 – 22 – 132

= – 176

Minor of an element:

Minor of an element is a determinant obtained by deleting the row and column in which that element occurs

Example:

Let $|A| = \begin{vmatrix} 1 & -1 & 3 \\ 0 & 4 & 2 \\ 11 & 5 & -3 \end{vmatrix}$

Minor of –1 = $\begin{vmatrix} 0 & 2 \\ 11 & -3 \end{vmatrix}$ = (0 – 22) = –22

Minor of 0 = $\begin{vmatrix} -1 & 3 \\ 5 & -3 \end{vmatrix}$ = (3 – 15) = –12

Co-factor of an element:

Minor of an element with proper sign attached is called co-factor of that element.

Example: Let $|A| = \begin{vmatrix} 3 & -2 & 1 \\ 2 & 0 & -3 \\ 4 & 5 & 11 \end{vmatrix}$

Co –factor of –2 = – $\begin{vmatrix} 2 & -3 \\ 4 & 11 \end{vmatrix}$ = –[(22) + 12]

= –34

Co –factor of 0 = + $\begin{vmatrix} 3 & 1 \\ 4 & 11 \end{vmatrix}$ = + (33 – 4)

= 29

Note: The sign for the element a_{ij} is $(-1)^{i+j}$

Properties of determinant: (not for examination)

Property (1):

The value of the determinant is unaltered by changing rows into columns and column into rows

(i.e), $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Property (2):

If any two rows or columns of a determinant are interchanged then the value of the determinant is changed in its sign only.i.e., $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ [Here $R_1 \leftrightarrow R_2$]

[Generally the rows and columns are denoted by R_1 , R_2 ..... and C_1 , C_2 ..... respectively]

Property (3):

If any two rows or columns of a determinant are identical or same, then the value of the determinant is zero.

(i.e) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$, (Here $R_1 \equiv R_2$)

Property (4):

If each element of a row or column of a determinant is multiplied by any constant $k \neq 0$, then the value of the determinant is multiplied by same constant k.

Property (5):

If each element of a row or column is expressed as the sum of two elements then the determinant can be expressed as the sum of two determinant of the same order.

(i.e) $\begin{vmatrix} a_1+d_1 & b_1+d_2 & c_1+d_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & d_2 & d_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Property (6):

Any determinant is unaltered when each elements of any row or column is added by the equimultiples of any parallel row or column.

(i.e) $\begin{vmatrix} a_1+ka_2 & b_1+kb_2 & c_1+kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ka_2 & kb_2 & kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k(0)$ (since $R_1 \equiv R_2$)

Property (7):

In a given determinant if two rows or column are identical for a = b, then (a – b) is a factor of the determinant.

(i.e) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \; for\ a = b$

$= \begin{vmatrix} 1 & 1 & 1 \\ b & b & c \\ b^2 & b^2 & c^2 \end{vmatrix} = 0$ ($c_1 \equiv c_2$)

(a-b) is a factor

Solution of simultaneous equations using Cramer's rule:

Consider the linear equations $a_1 x + b_1 y = c_1$

$a_2 x + b_2 y = c_2$

Let $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$; $\Delta_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and $\Delta_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

Then $x = \frac{\Delta_x}{\Delta}$; $y = \frac{\Delta y}{\Delta}$, Provided $\Delta \neq 0$. x, y are unique solutions of the given equations. This method of solving the linear equations is called Cramer's rule.

Similarly, for a set of three simultaneous linear equations in x, y and z.

$a_1 x + b_1 y + c_1 z = d_1$

$a_2 x + b_2 y + c_2 z = d_2$

$a_3 x + b_3 y + c_3 z = d_3$

The solution of the system of equations by Cramer's rule is given by

$$x = \frac{\Delta_x}{\Delta}, y = \frac{\Delta_y}{\Delta}, z = \frac{\Delta_z}{\Delta}$$

provided $\Delta \neq 0$. Where Δ, Δx, Δy and Δz are the determinents formed in the same way as defined above.

WORKED EXAMPLES

PART – A

1. Solve $\begin{vmatrix} x & 4 \\ 9 & x \end{vmatrix} = 0$

Solution:

$\begin{vmatrix} \boldsymbol{x} & \mathbf{4} \\ \mathbf{9} & \boldsymbol{x} \end{vmatrix} = \mathbf{0}$ By expanding we have

$x^2 - 36 = 0$

i.e $x^2 = 36$

$\therefore\ x = \pm 6$

2. Solve $\begin{vmatrix} x & 2 \\ 4 & 2 \end{vmatrix} = 0$

Solution*:*

$\begin{vmatrix} x & 2 \\ 4 & 2 \end{vmatrix} = 0$

Expand $2x - 8 = 0$

$2x = 8$

$x = 4$

3. Find the co-factor of 3 in the determinant $\begin{vmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \\ 5 & 6 & 7 \end{vmatrix}$

Solution:

Cofactor of 3 = $A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 0 \\ 5 & 7 \end{vmatrix}$

$= (-1)^4\ [7 - 0]$

$= +7$

4. Write down the minor of 3 in the determinant $\begin{vmatrix} 1 & -2 & -1 \\ 3 & 4 & -4 \\ 5 & 0 & 2 \end{vmatrix}$

Solution:

The minor of 3 $= \begin{vmatrix} -2 & -1 \\ 0 & 2 \end{vmatrix}$

$= -4 - 0 = -4$

PART – B

1. Find the value of 'm' when $\begin{vmatrix} m & 2 & 1 \\ 3 & 4 & 2 \\ -7 & 3 & 0 \end{vmatrix} = 0$

Solution:

Given $\begin{vmatrix} \boldsymbol{m} & \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{4} & \mathbf{2} \\ \mathbf{-7} & \mathbf{3} & \mathbf{0} \end{vmatrix} = \mathbf{0}$

Expand along R_1, we have

$$m(0-6) - 2(0+14) + 1(9+28) = 0$$
$$-6m - 28 + 37 = 0$$
$$-6m + 9 = 0$$
$$-6m = -9$$
$$6m = 9$$
$$m = \frac{9}{6} = \frac{3}{2}$$

2. Using Cramer's rule, solve the following equations $2x - 3y = 5$ and $x - 4y = 8$.

Solution:

$2x - 3y = 5$

$x - 4y = 8.$

where $\Delta = \begin{vmatrix} 2 & -3 \\ 1 & -4 \end{vmatrix}$

$= -8 + 3 = -5$

$\Delta x = \begin{vmatrix} 5 & -3 \\ 8 & -4 \end{vmatrix}$

$= -20 + 24 = 4$

$\Delta y = \begin{vmatrix} 2 & 5 \\ 1 & 8 \end{vmatrix}$

$= 16 - 5 = 11$

By Cramer's rule

$$x = \frac{\Delta_x}{\Delta} = \frac{4}{-5}; \quad y = \frac{\Delta y}{\Delta} = \frac{11}{-5}$$

3. Find the values of 'x' when $\begin{vmatrix} x & 6 & 3 \\ 1 & x & 1 \\ -2 & 4 & x \end{vmatrix} = 0$

Solution:

$$\begin{vmatrix} x & 6 & 3 \\ 1 & x & 1 \\ -2 & 4 & x \end{vmatrix} = 0$$

$$x(x^2 - 4) - 6(x + 2) + 3(4 + 2x) = 0$$
$$x^3 - 4x - 6x - 12 + 12 + 6x = 0$$
$$x^3 - 4x = 0$$
$$x(x^2 - 4) = 0$$
$$x = 0 \text{ and } x^2 - 4 = 0$$
$$x = 0 \text{ and } x^2 = 4$$
$$x = 0 \text{ and } x = \pm 2$$

PART –C

1. Using Cramer's rule, solve the following simultaneous equations,

$x + y + z = 2$

$2x - y - 2z = -1$

$x - 2y - z = 1$

Solution: Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & -1 \end{vmatrix}$

$= 1\,(1 - 4) - 1\,(-2 + 2) + 1\,(-4 + 1)$

$= -3 - 3 = -6$

$\Delta x = \begin{vmatrix} 2 & 1 & 1 \\ -1 & -1 & -2 \\ 1 & -2 & -1 \end{vmatrix}$

$= 2(1-4) - 1(1+2) + 1(2+1)$

$= 2\,(-3) - 1\,(3) + 1(3)$

$= -6 - 3 + 3$

$\Delta x = -6$

$\Delta y = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & -2 \\ 1 & 1 & -1 \end{vmatrix}$

$= 1\,(1 + 2) - 2\,(-2 + 2) + 1\,(2 + 1)$

$= 1\,(3) - 2\,(0) + 1(3)$

$= 3 + 3$

$\Delta y = 6$

$\Delta z = \begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & -1 \\ 1 & -2 & 1 \end{vmatrix} = 1\,(-1 - 2) - 1\,(2 + 1) + 2\,(-4 + 1)$

$= 1\,(-3) - 1\,(3) + 2(-3)$

$= -3 - 3 - 6$

$\Delta z = -12$

$x = \frac{\Delta x}{\Delta} = \frac{-6}{-6} = 1\,;\; y = \frac{\Delta y}{\Delta} = \frac{6}{-6} = -1$

$z = \frac{\Delta z}{\Delta} = \frac{-12}{-6} = 2$

2. Using Cramer's rule, solve the following simultaneous equations,

$x + y + z = 6$

$x - y + z = 2$

$2x + y - z = 1$

Solution:

Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$

$= 1\,(1 - 1) - 1\,(-1 - 2) + 1\,(1 + 2)$

$= 0 + 3 + 3 = 6$

$\Delta x = \begin{vmatrix} 6 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

$= 6\,(1 - 1) - 1\,(-2 - 1) + 1(2 + 1)$

$= 0 + 3 + 3$

$\Delta x = 6$

$\Delta y = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 1\,(-2 - 1) - 6\,(-1 - 2) + 1\,(1 - 4)$

$= 1\,(-3) - 6\,(-3) + 1(-3)$

$= -3 + 18 - 3$

$\Delta y = 12$

$\Delta z = \begin{vmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{vmatrix} = 1\,(-1 - 2) - 1\,(1 - 4) + 6\,(1 + 2)$

$= 1\,(-3) - 1\,(-3) + 6(3)$

$= -3 + 3 + 18$

$\Delta z = 18$

$x = \frac{\Delta_x}{\Delta} = \frac{6}{6} = 1$;

$y = \frac{\Delta y}{\Delta} = \frac{12}{6} = 2$;

$z = \frac{\Delta z}{\Delta} = \frac{18}{6} = 3$

Exercise

PART-A

1. Solve: $\begin{vmatrix} x & 9 \\ 4 & x \end{vmatrix} = 0$
2. Solve: $\begin{vmatrix} x & 2 \\ 18 & 3 \end{vmatrix} = 0$

PART-B

1. Find the value of 'm', when $\begin{vmatrix} 3 & m & 1 \\ 2 & 6 & 2 \\ 2 & 3 & 1 \end{vmatrix} = 0$

2. Find the value of ‘m’ ,when $\begin{vmatrix} m & 2 & -2 \\ 3 & 4 & 2 \\ 7 & 1 & 1 \end{vmatrix} = 0$
3. Using Cramer’s rule, solve the following simultaneous linear equation $x - 3y = 2$; $2x + 4y = 4$.
4. Using Cramer’s rule, solve the following simultaneous linear equation $2x - y = 5$; $x - y = 8$.

PART-C

1. Solve the following simultaneous equations, by using Cramer’s rule:
   $x + y + z = 3$; $4x - 3y + z = 2$; $-7x + 3y + 3z = -1$
2. Solve the following simultaneous equations, by using Cramer’s rule:
   $3x - y + 2z = 8$; $x - y + z = 2$; $2x + y - z = 1$
3. Solve the following simultaneous equations, by using Cramer’s rule:
   $2x + 2y - z = -3$; $3x + y + z = 4$; $x + y + 2z = 6$
4. Solve the following simultaneous equations, by using Cramer’s rule:
   $x + y + z = 3$; $2x + y - z = 2$; $3x - 2y - 2z = -1$

1.2. MATRICES

Introduction

The term matrix was first introduced by a French mathematician CAYLEY in the year 1857. The theory of matrices is one of the powerful tools of mathematics not only in the field of higher mathematics but also in other branches such as applied sciences, nuclear physics, statistics, economics and electrical circuits.

Definition:

A matrix is a rectangular array of elements arranged in rows and columns enclosed by brackets.

Example: $A = \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -7 & 5 \\ 3 & 6 & 5 \\ 9 & 1 & 2 \end{pmatrix}$

Order of a matrix:

If there are ‘m’ rows and ‘n’ columns in a matrix, then the order of the matrix is $m \times n$

Types of matrices:

(1) Row matrix:

A matrix having only one row and any number of columns is called a row matrix.

Example: $A = (1 \;\; 2 - 3)$

(2) Column matrix:

A matrix having only one column and any number of rows is called a column matrix.

Example: $A = \begin{bmatrix} 7 \\ 5 \\ 3 \end{bmatrix}$

(3) Square matrix:

A matrix which has equal number of rows and columns is called a square matrix.

Example: $A = \begin{bmatrix} 5 & -9 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 9 & 1 \\ 6 & 2 & -3 \\ 4 & 5 & 6 \end{bmatrix}$

(4) Null matrix (or) Zero matrix:

If all the elements of a matrix are zero, the matrix is called zero or null matrix.

Example : $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(5) Diagonal matrix:

A square matrix with all the elements equal to zero except those in the leading diagonal is called a diagonal matrix.

Example: $D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(6) Unit matrix:

Unit matrix is a square matrix in which the diagonal elements are all one and all the other elements are zeros.

Example: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(7) Determinant of a matrix:

Let 'A' be a square matrix. The determinant formed by the elements of A is said to be the determinant of matrix A. This is denoted by $|A|$.

(8) Singular and Non-Singular matrix:

A square matrix 'A' is called a singular matrix if $|A| = 0$. If $|A| \neq 0$, then the matrix A is called a non-singular matrix.

(9) Transpose of a matrix:

If the rows and columns of a matrix are interchanged, then the resultant matrix is called the transpose of the given matrix. It is denoted by A^T.

Example: if $A = \begin{pmatrix} 7 & -4 & 3 \\ 6 & 7 & 1 \\ 9 & 8 & 2 \end{pmatrix}$ then $A^T = \begin{pmatrix} 7 & 6 & 9 \\ -4 & 7 & 8 \\ 3 & 1 & 2 \end{pmatrix}$ and if $B = \begin{bmatrix} 3 & 5 \\ 2 & 8 \end{bmatrix}$ then $B^T = \begin{bmatrix} 3 & 2 \\ 5 & 8 \end{bmatrix}$

Adjoint (or) Adjugate matrix:

The transpose of the co-factor matrix is called the adjoint matrix or adjugate matrix. It is denoted by Adj.A

Inverse of a matrix:

Let A be a non-singular square matrix. If there exists a square matrix B, such that AB = BA = I, where I is the unit matrix of the same order as that of A, then B is called the inverse of A and it is denoted by A^{-1}. This can be determined by using the formula

$$A^{-1} = \frac{1}{|A|}\text{adj}A$$

If | A | = 0, then there is no inverse for the matrix A.

Rank of a matrix:

Let 'A' be any m × n matrix. It has square sub-matrices of different orders. The order of the largest square submtrix of A whose determinant has a non-zero value is known as the rank of the matrix A. Generally the rank is denoted by ρ(A), ρ(A) is always minimum of m and n.

Note:

The rank of a matrix is said to be r if

1) It has atleast one non zero minor of order r.
2) Every minor of A of order higher than r is zero.

Example:

Find the rank of the matrix $\begin{bmatrix} 1 & 5 & 6 \\ 2 & 3 & 4 \\ -1 & 2 & 2 \end{bmatrix}$

Let A = $\begin{bmatrix} 1 & 5 & 6 \\ 2 & 3 & 4 \\ -1 & 2 & 2 \end{bmatrix}$

$\rho(A) \leq \min\{3,3\}$

$\therefore\ \rho\ (A) \leq 3$

Now , $|A| = 1\begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} - 5\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} + 6\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix}$

$= 1(6-8) - 5(4+4) + 6(4+3)$

= 1(-2)-5(8)+6(7)

= -2 -40 +42 = 0

$\therefore \rho(A) \neq 3$

Consider the determinant of second order

$$\begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix} = (3-10) = -7 \neq 0$$

Hence, the rank of the matrix is $\rho(A) = 2$

WORKED EXAMPLE
PART A

1. Find the rank of the matrix $\begin{pmatrix} 3 & -4 \\ -6 & 8 \end{pmatrix}$

Solution

Let A = $\begin{pmatrix} 3 & -4 \\ -6 & 8 \end{pmatrix}$

$\rho(A) \leq \min\{2,2\}$

$\therefore\ \rho\ (A) \leq 2$

Now, $|A|$= $\begin{vmatrix} 3 & -4 \\ -6 & 8 \end{vmatrix}$

= (24 - 24) = 0

$\therefore \rho(A) \neq 2$

Since each element of A is non zero,

ρ(A) = 1

2. Find the adjoint of the matrix $\begin{pmatrix} 1 & 4 \\ -6 & 5 \end{pmatrix}$

Solution:

A = $\begin{pmatrix} 1 & 4 \\ -6 & 5 \end{pmatrix}$

Adj A = $\begin{pmatrix} 5 & -4 \\ 6 & 1 \end{pmatrix}$

3. Find the rank of $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

Solution:

Let A = $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

$\rho(A) \leq \min\{2,3\}$

$\therefore\ \rho\ (A) \leq 2$

Consider, $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$= 5 – 8 = – 3 ≠ 0

∴ ρ (A) = 2

PART-B

1. Find the inverse of $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$

Solution:

Let A= $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$

$\therefore\ |A| = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$

= 10 – 12 = –2 ≠ 0 $\quad \therefore A^{-1}$exist

Adj A = $\begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix}$

$A^{-1} = \frac{1}{|A|} adjA \quad \therefore A^{-1} = \frac{-1}{2}\begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix}$

2. Find the inverse of the matrix $\begin{pmatrix} 1 & 4 \\ 3 & -2 \end{pmatrix}$

Solution:

Let A= $\begin{pmatrix} 1 & 4 \\ 3 & -2 \end{pmatrix}$

$\therefore$ $|A| = \begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix}$

= -2 – 12 = –14 $\neq 0 \therefore A^{-1}$exist

Adj A = $\begin{pmatrix} -2 & -4 \\ -3 & 1 \end{pmatrix}$

$$A^{-1} = \frac{1}{|A|} adjA$$

$$A^{-1} = -\frac{1}{14}\begin{pmatrix} -2 & -4 \\ -3 & 1 \end{pmatrix}$$

3. Show that the matrix $\begin{pmatrix} 1 & 3 & 4 \\ 3 & 9 & 12 \\ 1 & 2 & 1 \end{pmatrix}$ is singular

Solution:

A = $\begin{pmatrix} 1 & 3 & 4 \\ 3 & 9 & 12 \\ 1 & 2 & 1 \end{pmatrix}$

$|A| = \begin{vmatrix} 1 & 3 & 4 \\ 3 & 9 & 12 \\ 1 & 2 & 1 \end{vmatrix}$

= $1\begin{vmatrix} 9 & 12 \\ 2 & 1 \end{vmatrix} - 3\begin{vmatrix} 3 & 12 \\ 1 & 1 \end{vmatrix} + 4\begin{vmatrix} 3 & 9 \\ 1 & 2 \end{vmatrix}$

= 1(9 - 24) – 3 (3-12) + 4(6 – 9)

= - 15+27-12

= 0

Hence, the given matrix is singular

4. Find the rank of the matrix $\begin{pmatrix} 5 & -15 \\ -6 & 18 \\ 7 & -21 \end{pmatrix}$

Solution:

Let A = $\begin{pmatrix} 5 & -15 \\ -6 & 18 \\ 7 & -21 \end{pmatrix}$

$\rho(A) \leq \min\{3,2\}$

$\therefore$ $\rho\ (A) \leq 2$

$\begin{vmatrix} 5 & -15 \\ -6 & 18 \end{vmatrix}$ = (90-90) = 0

$\begin{vmatrix} 5 & -15 \\ 7 & -21 \end{vmatrix}$ = -105+105 = 0

$\begin{vmatrix} -6 & 18 \\ 7 & -21 \end{vmatrix}$ = 126-126 = 0

$\therefore \rho(A) \neq 2$

Hence $\rho(A) = 1$

5. Find the rank of the matrix $\begin{pmatrix} 2 & -15 \\ 3 & 10 \\ 3 & -1 \end{pmatrix}$

Solution:

Let A = $\begin{pmatrix} 2 & -15 \\ 3 & 10 \\ 3 & -1 \end{pmatrix}$

$\rho(A) \leq \min\{3,2\}$

$\therefore\ \rho\ (A) \leq 2$

$\begin{vmatrix} 2 & -15 \\ 3 & 10 \end{vmatrix}$ = (20 + 45) = 65 $\neq$ 0

$\rho(A) = 2$

PART-C

1. Find the inverse of the matrix $\begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{pmatrix}$

Solution:

Let A = $\begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{pmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{vmatrix}$$

= 1 (3 –0) – 1 (6 – 0) – 1 (4 + 1) = 1 (3) – 1 (6) – 1 (5)

= 3 – 6 – 5 = – 8 $\neq$ 0

$\therefore$ A^{-1} exists

Cofactor of a_{11} = A_{11} = $+\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix}$ = 3 – 0 = 3

$A_{12} = -\begin{vmatrix} 2 & 0 \\ -1 & 3 \end{vmatrix}$ = – (6–0) = – 6

$A_{13} = +\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}$ = 4+1= 5

$A_{21} = -\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}$ = - (3+2) = - 5

$A_{22} = + \begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} = 3 - 1 = 2$

$A_{23} = - \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = -(2+1) = -3$

$A_{31} = + \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 0+1 = 1$

$A_{32} = - \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = -(0+2) = -2$

$A_{33} = + \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1 - 2 = -1$

Cofactor matrix of A = $\begin{pmatrix} 3 & -6 & 5 \\ -5 & 2 & -3 \\ 1 & -2 & -1 \end{pmatrix}$

Adj A = (cofactor matrix)^T = $\begin{pmatrix} 3 & -6 & 5 \\ -5 & 2 & -3 \\ 1 & -2 & -1 \end{pmatrix}^T$

$$= \begin{pmatrix} 3 & -5 & 1 \\ -6 & 2 & -2 \\ 5 & -3 & -1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA$$

$$A^{-1} = \frac{-1}{8} \begin{pmatrix} 3 & -5 & 1 \\ -6 & 2 & -2 \\ 5 & -3 & -1 \end{pmatrix}$$

2. Find the inverse of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$

Solution:

Let A = $\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 2 & 0 & -1 \end{vmatrix}$$

= 1 (1 – 0) – 1 (-2 – 0) + 1 (0 + 2) = 1 (1) – 1 (-2) + 1 (2)
= 1 +2 +2 = 5

Co-factors of a_{11} of = $A_{11} = + \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1 - 0 = 1$

$A_{12} = - \begin{vmatrix} 2 & 0 \\ 2 & -1 \end{vmatrix} = -(-2-0) = 2$

$A_{13} = + \begin{vmatrix} 2 & -1 \\ 2 & 0 \end{vmatrix} = 0+2 = 2$

$A_{21} = -\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix}$ = - (-1+0) = 1

$A_{22} = +\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}$ = -1 -2 = -3

$A_{23} = -\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}$ = - (0-2) = 2

$A_{31} = \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix}$ = 0+1 = 1

$A_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}$ = - (0-2) = 2

$A_{33} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}$ = –1 – 2 = –3

Cofactor matrix of A = $\begin{pmatrix} 1 & 2 & 2 \\ 1 & -3 & 2 \\ 1 & 2 & -3 \end{pmatrix}$

Adj A = (cofactor matrix)^T = $\begin{pmatrix} 1 & 2 & 2 \\ 1 & -3 & 2 \\ 1 & 2 & -3 \end{pmatrix}^T$

$= \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{pmatrix}$

$$A^{-1} = \frac{1}{|A|} adjA$$

$$\therefore A^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{pmatrix}$$

3. Find the inverse of the matrix $\begin{pmatrix} -1 & 2 & 6 \\ 0 & 7 & 2 \\ -1 & 1 & 4 \end{pmatrix}$

Solution:

Let A = $\begin{pmatrix} -1 & 2 & 6 \\ 0 & 7 & 2 \\ -1 & 1 & 4 \end{pmatrix}$

$$|A| = \begin{vmatrix} -1 & 2 & 6 \\ 0 & 7 & 2 \\ -1 & 1 & 4 \end{vmatrix}$$

= -1 (28 – 2) – 2 (0 +2) + 6 (0 + 7) = -1 (26) – 2 (2) + 6 (7)
= - 26 - 4 + 42 = 12

$\therefore A^{-1}$ exists

Co-factors of of a_{11} = A_{11} = $+\begin{vmatrix} 7 & 2 \\ 1 & 4 \end{vmatrix}$ = 28 – 2 = 26

$A_{12} = -\begin{vmatrix} 0 & 2 \\ -1 & 4 \end{vmatrix}$ = – (0+2) = –2

$A_{13} = +\begin{vmatrix} 0 & 7 \\ -1 & 1 \end{vmatrix} = 0+7= 7$

$A_{21} = -\begin{vmatrix} 2 & 6 \\ 1 & 4 \end{vmatrix} = -(8-6) = -2$

$A_{22} = +\begin{vmatrix} -1 & 6 \\ -1 & 4 \end{vmatrix} = -4+6 = 2$

$A_{23} = -\begin{vmatrix} -1 & 2 \\ -1 & 1 \end{vmatrix} = -(-1+2) = -1$

$A_{31} = +\begin{vmatrix} 2 & 6 \\ 7 & 2 \end{vmatrix} = 4 - 42 = -38$

$A_{32} = -\begin{vmatrix} -1 & 6 \\ 0 & 2 \end{vmatrix} = -(-2+0) = 2$

$A_{33} = +\begin{vmatrix} -1 & 2 \\ 0 & 7 \end{vmatrix} = -7 - 0 = -7$

Cofactor matrix of A = $\begin{pmatrix} 26 & -2 & 7 \\ -2 & 2 & -1 \\ -38 & 2 & -7 \end{pmatrix}$

Adj A = (cofactor matrix)^T = $\begin{pmatrix} 26 & -2 & 7 \\ -2 & 2 & -1 \\ -38 & 2 & -7 \end{pmatrix}^T$

$$= \begin{pmatrix} 26 & -2 & -38 \\ -2 & 2 & 2 \\ 7 & -1 & -7 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA$$

$$\therefore\ A^{-1} = \frac{1}{12}\begin{pmatrix} 26 & -2 & -38 \\ -2 & 2 & 2 \\ 7 & -1 & -7 \end{pmatrix}$$

4. Find the rank of the matrix $\begin{pmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Solution:

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\rho(A) \leq \min\{3,3\}$$

$$\therefore\ \rho\ (A) \leq 3$$

$$|A| = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\ (1-1) + 1\ (1-1) + 3\ (1-1) = 0$$

$$\therefore \rho(A) \neq 3$$

Now $|A| = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$

$= 2 + 1 = 3 \neq 0$

$\rho(A) = 2$

5. Find the rank of the matrix $\begin{pmatrix} 1 & -2 & 3 \\ -2 & -4 & -6 \\ 5 & 1 & -1 \end{pmatrix}$

Solution:

$$A = \begin{pmatrix} 1 & -2 & 3 \\ -2 & -4 & -6 \\ 5 & 1 & -1 \end{pmatrix}$$

$\rho(A) \leq \min\{3,3\}$

$\therefore\ \rho\ (A) \leq 3$

$$|A| = \begin{vmatrix} 1 & -2 & 3 \\ -2 & -4 & -6 \\ 5 & 1 & -1 \end{vmatrix}$$

$= 1\ (4 + 6) + 2\ (2 + 30) + 3\ (-2 + 20) = 128 \neq 0$

$\rho(A) = 3$

6. Find the rank of the Matrix $\begin{pmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{pmatrix}$

Solution:

$$A = \begin{pmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{pmatrix}$$

$\rho(A) \leq \min\{3,4\}$

$\therefore\ \rho\ (A) \leq 3$

$$\begin{vmatrix} 2 & 3 & 4 \\ 5 & 2 & 0 \\ -4 & 5 & 12 \end{vmatrix} = 2(24 - 0) - 3(60 - 0) + 4(25 + 8)$$

$= 48 - 180 + 132 = 0$

$$\begin{vmatrix} 3 & 4 & -1 \\ 2 & 0 & -1 \\ 5 & 12 & -1 \end{vmatrix} = (0 + 12) - 4(-2 + 5) - 1(24 - 0)$$

$= 12 + 12 - 24 = 0$

Similarly we can prove $\begin{vmatrix} 2 & 4 & -1 \\ 5 & 0 & -1 \\ -4 & 12 & -1 \end{vmatrix} = 0$ and $\begin{vmatrix} 2 & 3 & -1 \\ 5 & 2 & -1 \\ -4 & 5 & -1 \end{vmatrix} = 0$

$\therefore \rho(A) \neq 3$

$$\begin{vmatrix} 2 & 3 \\ 5 & 2 \end{vmatrix} = 4 - 15 = -11 \neq 0$$

$\therefore \rho(A) = 2$

7. Find the rank of the Matrix $\begin{pmatrix} -1 & -3 & -2 & 3 \\ -2 & -6 & -4 & 6 \\ -3 & -9 & -6 & 9 \end{pmatrix}$

Solution:

$$A=\begin{pmatrix}-1 & -3 & -2 & 3\\ -2 & -6 & -4 & 6\\ -3 & -9 & -6 & 9\end{pmatrix}$$

$$\rho(A) \le \min\{3,4\}$$

$$\therefore\ \rho\ (A) \le 3$$

$$\begin{vmatrix}-1 & -3 & -2\\ -2 & -6 & -4\\ -3 & -9 & -6\end{vmatrix} = -1(36-36)+3(12-12)-2(18-18)= 0$$

$$\begin{vmatrix}-1 & -3 & 3\\ -2 & -6 & 6\\ -3 & -9 & 9\end{vmatrix} = -1(-54+54)+3(-18+18)+3(18-18) = 0$$

Similarly, we can prove $\begin{vmatrix}-1 & -2 & 3\\ -2 & -4 & 6\\ -3 & -6 & 9\end{vmatrix} = 0$ and $\begin{vmatrix}-3 & -2 & 3\\ -6 & -4 & 6\\ -9 & -6 & 9\end{vmatrix} = 0$

$$\therefore \rho(A) \ne 3$$

$$\begin{vmatrix}-1 & -3\\ -2 & -6\end{vmatrix} = 6-6 = 0$$

$$\begin{vmatrix}-2 & 3\\ -4 & 6\end{vmatrix} = 12-12 = 0$$

$$\begin{vmatrix}-2 & -6\\ -3 & -9\end{vmatrix} = 18-18 = 0$$

$$\begin{vmatrix}-4 & 6\\ -6 & 9\end{vmatrix} = 36-36 = 0$$

Similarly, we can prove the remaining second order determinant value is zero

$$\therefore \rho(A) = 1.$$

BINOMIALTHEOREM

Binomial Theorem for rational index:

If x is numerically less than one and n any rational number, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \infty$$

NOTE

(i) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots\ldots\ldots$

(ii) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots\ldots\ldots$

(iii) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots\ldots$

(iv) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots\ldots\ldots$

WORKED EXAMPLES

PART –A

1. Write down the expansion for $(1-x)^{-1}$.

   Solution:

   $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots\ldots\ldots$

2. Find the value of $(1+2x)^{-2}$.

   Solution:

   $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots\ldots\ldots$, Put $x = 2x$

$\therefore (1 + 2x)^{-2} = 1 - 2(2x) + 3(2x)^2 - 4(2x)^3 + ...$

$= 1 - 4x + 12x^2 - 32x^3 + ...$

3. Find the value of $(1 - 2x)^{-2}$.

Solution:

$(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ...,$ put $x = 2x$

$\therefore (1 - 2x)^{-2} = 1 + 2(2x) + 3(2x)^2 + 4(2x)^3 + ...$

$= 1 + 4x + 12x^2 + 32x^3 + ...$

4. Find the value of $(1 + 2x)^{-1}$.

Solution:

$(1 + x)^{-1} = 1 - x + x^2 - x^3 + ...,$ put $x = 2x$

$\therefore (1 + 2x)^{-2} = 1 - (2x) + (2x)^2 - (2x)^3 + ...$

$= 1 - 2x + 4x^2 - 8x^3 + ...$

5. Find the value of $(1 - 2x)^{-1}$.

Solution:

$(1 - x)^{-1} = 1 + x + x^2 + x^3 + ...,$ put $x = 2x$

$\therefore (1 - 2x)^{-1} = 1 + (2x) + (2x)^2 + (2x)^3 + ...$

$= 1 + 2x + 4x^2 + 8x^3 + ...$

6. Find the value of $(1 + 3x)^{-2}$.

Solution:

$(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ...,$ put $x = 3x$

$\therefore (1 + 3x)^{-2} = 1 - 2(3x) + 3(3x)^2 - 4(3x)^3 + ...$

$= 1 - 6x + 27x^2 - 108x^3 + ...$

Exercise

PART-A

1. Find the adjoint of the matrix $\begin{pmatrix} 2 & -4 \\ 3 & 1 \end{pmatrix}$
2. Find the rank of the matrix $\begin{pmatrix} 1 & 4 \\ -6 & 5 \end{pmatrix}$
3. Find the adjoint of the matrix $\begin{pmatrix} 11 & -1 \\ 3 & -5 \end{pmatrix}$

PART-B

1. Find the inverse of $\begin{pmatrix} -2 & 3 \\ 1 & 5 \end{pmatrix}$
2. Find the inverse of $\begin{pmatrix} 1 & -3 \\ 4 & 2 \end{pmatrix}$
3. Find the value of $(1 + x^2)^{-1}$

4. Find the value of $(1 - x^2)^{-1}$

PART-C

1. Find the inverse of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 7 & 2 \\ 1 & 1 & 4 \end{pmatrix}$
2. Find the inverse of the matrix $\begin{pmatrix} 3 & 6 & -5 \\ 7 & 9 & -2 \\ -4 & -6 & 7 \end{pmatrix}$
3. Find the rank of the Matrix $\begin{pmatrix} 1 & 2 & 3 & -1 \\ 2 & 4 & 6 & -2 \\ 3 & 6 & 9 & -3 \end{pmatrix}$
4. Find the rank of the Matrix $\begin{pmatrix} 2 & 3 & 4 & 1 \\ 5 & 2 & 0 & 1 \\ -4 & 5 & 12 & 1 \end{pmatrix}$
5. Find the rank of the matrix $\begin{pmatrix} 1 & 2 \\ 3 & -8 \\ -4 & 5 \end{pmatrix}$
6. Find the rank of the matrix $\begin{pmatrix} 3 & -4 & 1 \\ -6 & 8 & -2 \end{pmatrix}$

1.3 ALGEBRA OF THE COMPLEX NUMBERS

Introduction:

Let us consider the quadratic equation $ax^2 + bx + c = 0$. The solution of this equation is given by the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ which is meaningful only when $b^2 - 4ac > 0$. Because the square root of a real number is always positive and it cannot be negative. If it is negative, then the solution for the equation extends the real number system to a new kind of number system that allows the square root of negative numbers. The square root of – 1 is denoted by the symbol i, called the imaginary unit, which was first introduced in mathematics by the famous Swiss mathematician, Leonhard Euler in 1748. Thus for any two real numbers a and b, we can form a new number a + ib is called a **complex number**. The set of all complex numbers denoted by C and the nomenclature of a complex number was introduced by a German mathematician C.F. Gauss.

Definition: Complex Number

A number which is of the form a + ib where a, b $\in$ R and $i = \sqrt{-1}$ is called a complex number and it is denoted by z. If z = a + ib then *a* is called the real part of z and *b* is called the imaginary part of z and are denoted by Re(z) and Im(z).

For example, if z = 3 + 4i then Re (z) = 3 and Im (z) = 4.

Note:

In the complex number z = a + ib we have,

(i) If a = 0, then z is purely real.

(i) If b = 0, then z is purely imaginary.

(ii) z = a + ib = (a, b) (i.e) any complex number can be expressed as an ordered pair.

Conjugate of a complex number:

If z = a + ib, then the conjugate of z is defined by a – ib and it isdenoted by $\overline{z}$.

Thus, if z = a + ib, then $\overline{z}$ = a – ib.

Results:

i. $\bar{\bar{z}} = z$

ii. a = Re (z) = $\frac{z+\bar{z}}{2}$ & b = Im(z) = $\frac{z-\bar{z}}{2}$

iii. $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$

iv. $\overline{z_1 - z_2} = \bar{z_1} - \bar{z_2}$

v. $\overline{z_1 z_2} = \bar{z_1}\,\bar{z_2}$

vi. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$, where $z_2 \neq 0$

vii. $\overline{z^n} = \bar{z}^n$

Algebra of complex numbers:

(i) Addition of two complex numbers:

Let z_1 = a +ib and z_2 = c + id be any two complex numbers then their sum is defined as

$z_1 + z_2$ = a + ib + c + id = (a + c) + i (b + d)

Note:

z + $\bar{z}$ = 2a (Real number).

(ii) Difference of two complex numbers:

Let z_1 = a +ib and z_2 = c + id be any two complex numbers then their difference is defined as

$z_1 - z_2$ = (a + ib) – (c + id) = (a – c) + i (b – d)

Note:

z – $\bar{z}$ = 2ib (Imaginary number).

(iii) Multiplication of two compelx numbers:

Let z_1 = a+ib and z_2 = c + id be any two complex numbers then their product is defined as,

$z_1 z_2$ = (a + ib) (c + id)

= a c + i ad + i bc + i^2 bd

$z_1.z_2$ = (ac – bd) + i (ad + bc)

Note:

z. $\bar{z}$ = (a + ib) (a –ib) = $a^2 + b^2$

(iv) Division of two complex numbers:

Let z_1 = a + ib and z_2 = c + id ≠ 0 be any two complex numbers, then their quotient is defined as

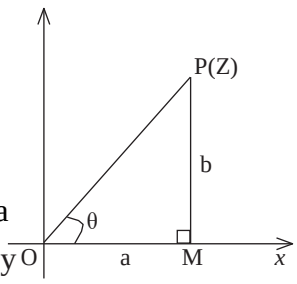
$$\frac{z_1}{z_2} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id} = \frac{[ac+bd]}{c^2+d^2} + i\frac{[bc-ad]}{c^2+d^2}$$

Modulus of a complex number:

If z = a + ib is a complex number then the modulus (or) absolute value of z is defined as $\sqrt{a^2+b^2}$ and is denoted by | z |. Thus, if z = a+ib then | z | = $\sqrt{a^2+b^2}$

Polar form of a Complex Number:

Let (r, θ) be the Polar co-ordinates of the point P representing the complex number z = a + ib. Then from the fig. we get,



$\cos\theta = \frac{OM}{OP} = \frac{a}{r}$ and $\sin\theta = \frac{PM}{OP} = \frac{b}{r}$

⇒ a = r cos θ and b = r sin θ

where $r = \sqrt{a^2 + b^2}$ =| a + ib| is called the **modulus** of z = a + ib. Also, tan θ = b/a

θ = $\tan^{-1}\left(\frac{b}{a}\right)$ is called the **amplitude** or **argument** of z = a + ib and denoted by amp(z) or arg(z)

arg(z) = θ = $\tan^{-1}\left(\frac{b}{a}\right)$ Hence z = a + ib = r (cos θ + i sin θ) is called the Polar form or the modulus amplitude form of the complex number

NOTE: Steps to find the amplitude of complex number

Step 1: Make $\sin\theta > 0$, $\cos\theta > 0$, then find an angle α that satisfies both $\sin\theta$ and $\cos\theta$

Step 2: if $\sin\theta > 0$ and $\cos\theta > 0$, then amplitude $\theta = \alpha$

Step 3: if $\sin\theta > 0$ and $\cos\theta < 0$, then amplitude $\theta = \pi - \alpha$

Step 4: if $\sin\theta < 0$ and $\cos\theta < 0$, then amplitude $\theta = -(\pi - \alpha)$

Step 5: if $\sin\theta < 0$ and $\cos\theta > 0$, then amplitude $\theta = -\alpha$

Theorems of Complex numbers:

1) The product of two complex numbers is a complex number whose modulus is the product of their modulii and whose amplitude is the sum of their amplitudes
   i.e, $|z_1 z_2| = |z_1|\,|z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
2) The quotient of two complex numbers is a complex number whose modulus is the quotient of their modulii and whose amplitude is the difference of their amplitudes
   ie, $\left|\frac{z_1}{z_2}\right| = \frac{|z_2|}{|z_2|}$ $and\ arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Euler's formula:

The symbol $e^{i\theta}$ is defined by $e^{i\theta} = \cos\theta + i\sin\theta$ is known as Euler's formula.If $z \neq 0$ then Z = r (cos θ+i sin θ) = $re^{i\theta}$. This is called the exponential form of the complex number z.

Multiplication and Division of complex numbers (Geometrical proof not for the exam)

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be any two complex numbers in Polar form then their product is given by

$$z_1 z_2 = r_1 r_2\,[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Also the division of the above two complex numbers is given by

$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + isin\,(\theta_1 - \theta_2)]$ where $z_2 \neq 0$

WORKED EXAMPLES

PART – A

1. Find the value of $i^2 + i^3 + i^4$

Solution:

$i^2 + i^3 + i^4$

= -1 + i + 1

= i

2. Find the value of $i^{44} + i^{45} + i^{46} + i^{47}$

Solution:

$i^{44} + i^{45} + i^{46} + i^{47}$

$= 1 + i + i^2 + i^3$

$= 1 + i - 1 - i$

$= 0$

3. Find the value of $i^5 + i^6 + i^7$

Solution:

$i^5 + i^6 + i^7$

$= i + i^2 + i^3$

$= i - 1 - i = -1$

4. If $z_1 = 2 + 3i$ and $z_2 = 4 - 5i$, find $z_1 + z_2$.

Solution:

Given: $z_1 = 2 + 3i$ & $z_2 = 4 - 5i$

$z_1 + z_2 = (2 + 3i) + (4 - 5i)$

$= 2 + 3i + 4 - 5i$

$= (2 + 4) + (3i - 5i)$

$\Rightarrow \boxed{z_1 + z_2 = 6 - 2i}$

5. If $z_1 = 3 - 4i$ and $z_2 = -2 + 3i$, find the value of $2z_1 - 3z_2$.

Solution:

Given: $z_1 = 3 - 4i$ and $z_2 = -2 + 3i$

$2z_1 - 3z_2 = 2(3 - 4i) - 3(-2 + 3i)$

$= 6 - 8i + 6 - 9i$

$\Rightarrow \boxed{2z_1 - 3z_2 = 12 - 17i}$

6. Express the complex number $(3 + 2i)(4 + 2i)$ in $a + ib$ form.

Solution:

$(3 + 2i)(4 + 2i) = 12 + 6i + 8i + 4i^2$

$= 12 + 14i - 4$

$= 8 + 14i = a + ib$ form

7. Find the modulus and amplitude of $1 + i$.

Solution:

Let $z = 1 + i$

Here $a = 1$ & $b = 1$

Modulus :$|z| = \sqrt{a^2 + b^2} = \sqrt{(1)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2}$

$\sin\theta = \frac{b}{r}$ and $\cos\theta = \frac{a}{r}$

$\therefore$ $\sin\theta = \frac{1}{\sqrt{2}}$ and $\cos\theta = \frac{1}{\sqrt{2}}$

Here, $\sin\theta > 0$, $\cos\theta > 0$

$\therefore$ Amplitude, $\theta = \frac{\pi}{4}$

PART - B

1. Express the complex number $\frac{1}{3-2i} + \frac{1}{2-3i}$ in a+ib form.

Solution:

Let Z = $\frac{1}{3-2i} + \frac{1}{2-3i}$

$= \frac{1}{3-2i} \times \frac{3+2i}{3+2i} + \frac{1}{2-3i} + \frac{2+3i}{2+3i}$

$= \frac{3+2i}{3^2+2^2} + \frac{2+3i}{2^2+3^2}$

$= \frac{3+2i+2+3i}{13}$

$= \frac{5+5i}{13}$

$= \frac{5}{13} + \frac{5i}{13}$ = a+ib form

2. Find the real and imaginary part of $\frac{1}{3+2i}$

Solution:

Let z = $\frac{1}{3+2i} = \frac{1}{3+2i} \times \frac{3-2i}{3-2i}$

$= \frac{3-2i}{3^2-(2i)^2}$

$= \frac{3-2i}{9+4}$

$= \frac{3-2i}{13}$

z = $\frac{3}{13} - \frac{2}{13}i$

$\therefore$ Re(z) = $\frac{3}{13}$ & Im(z) = $\frac{-2}{13}$

3. Find the conjugate of $\frac{1}{1+i}$

Solution:

Let z = $\frac{1}{1+i}$

$Z = \frac{1}{1+i} \times \frac{1-i}{1-i}$

$Z = \frac{1-i}{1-i^2}$

$Z = \frac{1-i}{1+1}$

$Z = \frac{1-i}{2}$

$Z = \frac{1}{2} - \frac{i}{2}$

$\therefore$ Conjugate of z = $\bar{z} = \frac{1}{2} + \frac{i}{2}$

4. Find the modulus and amplitude of $\frac{1}{2} + \frac{\sqrt{3}}{2}i$

Solution:

Let z = $\frac{1}{2} + \frac{\sqrt{3}}{2}i$

Here a = $\frac{1}{2}$ and b = $\frac{\sqrt{3}}{2}$

Modulus $|z| = r = \sqrt{a^2 + b^2}$

$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$r = \sqrt{\frac{1}{4} + \frac{3}{4}}$

$r = \sqrt{\frac{4}{4}} = 1$

$\sin\theta = \frac{b}{r}$ and $\cos\theta = \frac{a}{r}$

$\therefore$ $\sin\theta = \frac{\sqrt{3}}{2}$ and $\cos\theta = \frac{1}{2}$

Here, $\cos\theta > 0$, and $\sin\theta > 0$

$\therefore$ amplitude, $\theta = \frac{\pi}{3}$

5. Find the modulus and amplitude of $\frac{1}{2} - \frac{\sqrt{3}}{2}i$

Solution:

Let z = $\frac{1}{2} - \frac{\sqrt{3}}{2}i$

Here a = $\frac{1}{2}$ and b = $-\frac{\sqrt{3}}{2}$

Modulus, $|z| = r = \sqrt{a^2 + b^2}$

$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2}$

$$r = \sqrt{\frac{1}{4} + \frac{3}{4}}$$

$$r = \sqrt{\frac{4}{4}} = 1$$

$\sin\theta = \frac{b}{r}$ and $\cos\theta = \frac{a}{r}$

$\therefore \sin\theta = -\frac{\sqrt{3}}{2}$ and $\cos\theta = \frac{1}{2}$

Here, $\sin\theta < 0$ and $\cos\theta > 0$ therefore $\theta = -\alpha$

Let $\sin\theta = \frac{\sqrt{3}}{2}$ and $\cos\theta = \frac{1}{2}$

$\therefore \ \alpha = \frac{\pi}{3}$

$\therefore$ amplitude, $\theta = -\frac{\pi}{3}$

6. Find the modulus and amplitude of $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$

Solution:

Let $z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

Here $a = -\frac{1}{2}$ & $b = -\frac{\sqrt{3}}{2}$

Modulus $|z| = r = \sqrt{a^2 + b^2}$

$$r = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2}$$

$$r = \sqrt{\frac{1}{4} + \frac{3}{4}}$$

$$r = \sqrt{\frac{4}{4}} = 1$$

$\sin\theta = \frac{b}{r}$ and $\cos\theta = \frac{a}{r}$

$\therefore \sin\theta = -\frac{\sqrt{3}}{2}$ and $\cos\theta = -\frac{1}{2}$

Here, $\sin\theta < 0$, and $\cos\theta < 0 \therefore \theta = -(\pi - \alpha)$

Let $\sin\theta = \frac{\sqrt{3}}{2}$ and $\text{Cos}\,\theta = \frac{1}{2}$

$\therefore \ \alpha = \frac{\pi}{3}$

$\therefore\ \theta = -(\pi - \frac{\pi}{3})$

$\therefore$ amplitude, $\theta = \frac{-2\pi}{3}$

PART - C

1. Find the real and imaginary parts of the complex number $\frac{(1+i)(2-i)}{1+3i}$.

Solution:

Let z = $\frac{(1+i)(2-i)}{1+3i}$

$= \frac{2-i+2i-i^2}{1+3i}$

$= \frac{2+i+1}{1+3i} = \frac{3+i}{1+3i}$

$= \frac{3+i}{1+3i} \times \frac{1-3i}{1-3i}$

$= \frac{3-9i+i-3i^2}{1^2-(3i)^2}$

$= \frac{3-8i+3}{1+9}$

$= \frac{6-8i}{10}$

$= \frac{3-4i}{5}$

Z= $\frac{3}{4} - \frac{4}{5}i$ = a+ib form

Re(Z) = $\frac{3}{4}$ & Im(Z) = $-\frac{4}{5}$

2. Express the complex number $\frac{i-4}{3-2i} + \frac{4i+1}{2-3i}$ in a+ib form

Solution:

Let z = $\frac{i-4}{3-2i} + \frac{4i+1}{2-3i}$

$= \frac{i-4}{3-2i} \times \frac{3+2i}{3+2i} + \frac{4i+1}{2-3i} \times \frac{2+3i}{2+3i} = \frac{3i-12+2i^2-8i}{3^2+2^2} + \frac{8i+2+12i^2+3i}{2^2+3^2}$

$= \frac{-5i-14}{13} + \frac{11i-10}{13}$

$= \frac{6i-24}{13}$

$= \frac{6i-24}{13} = \frac{-24}{13} + \frac{6i}{13}$ = a+ib form

3. Find the modulus and amplitude of $\frac{1+3\sqrt{3}i}{\sqrt{3}+2i}$.

Solution:

Let Z = $\frac{1+3\sqrt{3}i}{\sqrt{3}+2i}$

$= \frac{1+3\sqrt{3}i}{\sqrt{3}+2i} \times \frac{\sqrt{3}-2i}{\sqrt{3}-2i}$

$= \frac{\sqrt{3}-2i+9i-6\sqrt{3}i^2}{(\sqrt{3})^2-(2i)^2}$

$= \frac{\sqrt{3}+7i+6\sqrt{3}}{3+4} = \frac{\sqrt{3}(1+6)+7i}{7} = \frac{7\sqrt{3}+7i}{7} = \frac{7(\sqrt{3}+i)}{7}$

$= \sqrt{3} + i = a + ib\ form$

Here a = $\sqrt{3}$ and b = 1

Modulus of Z, r = $\sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$

sin $\theta = \frac{b}{r}$ and cos $\theta = \frac{a}{r}$

$\therefore$ sin $\theta = \frac{1}{2}$ and cos $\theta = \frac{\sqrt{3}}{2}$

Here, sin $\theta > 0$, and cos $\theta > 0$

$\therefore$ amplitude, $\theta = \frac{\pi}{6}$

4. Find the modulus and argument of the complex number $\frac{1-i}{1+i}$.

Solution

Let Z = $\frac{1-i}{1+i} = \frac{1-i}{1+i} \times \frac{1-i}{1-i}$

$= \frac{1-i-i+i^2}{1^2-i^2}$

$= \frac{1-2i-1}{1+1}$

$= \frac{-2i}{2}$

Z = - i where a= 0 and b = -1

Modulus :$|z| = \sqrt{a^2+b^2} = \sqrt{(0)^2+(-1)^2} = \sqrt{1} = 1$

sin $\theta = \frac{b}{r}$ and cos $\theta = \frac{a}{r}$

$\therefore$ sin θ = -1 and cos θ = 0

Here, sin $\theta < 0$, and cos $\theta > 0$ therefore $\theta = -\alpha$

Let sin θ = 1 and cos θ = 0

$\therefore \alpha = \frac{\pi}{2}$

$\therefore$ amplitude $\theta = -\frac{\pi}{2}$

5. Find the modulus and argument of the complex number $\frac{5-i}{2-3i}$.

Solution:

Let Z = $\frac{5-i}{2-3i} = \frac{5-i}{2-3i} \times \frac{2+3i}{2+3i}$

$= \frac{10+15i-2i-3i^2}{2^2-(3i)^2}$

$= \frac{10+13i+3}{4+9}$

$= \frac{13+13i}{13}$

Z = 1+ i where a= 1 and b = 1

$|z| = r = \sqrt{1^2 + 1^2} = 1$

$\sin\theta = \frac{b}{r}$ and $\cos\theta = \frac{a}{r}$

$\therefore \sin\theta = 1$ and $\cos\theta = 1$

$\therefore \sin\theta > 0, \cos\theta > 0$

$\therefore$ amplitude $\theta = \frac{\pi}{4}$

Exercise

PART- A

1. Find the value of $i^{39} + i^{40} + i^{41} + i^{42}$
2. Find the value of $i^{30} + i^{29} + i^{28} + i^{27}$
3. Express; (4 - 2i) (5 + 7i) in a + ib form.
4. Express; (2 - 5i) (-2 - i) in a + ib form.
5. Find the modulus and amplitude of 1 + i.
6. Find the modulus and amplitude of -8 +16 i.

PART - B

1. Find the real and imaginary parts of $\frac{1}{3-6i}$
2. Find the real and imaginary parts of $\frac{1}{-8+2i}$

PART- C

1. Express the complex number $\frac{2i-9}{1-2i} + \frac{1}{2+3i}$ in a+ib form
2. Express the complex number $\frac{i-4}{4+2i} + \frac{i+1}{1+5i}$ in a+ib form
3. Express the complex number $\frac{1-4i}{3-2i} + \frac{2+7i}{2-3i}$ in a+ib form
4. Find the modulus and argument of the complex number $\frac{5-3i}{1+3i}$.
5. Find the modulus and argument of the complex number $\frac{5+4i}{3-4i}$
6. Find the modulus and argument of the complex number $\frac{i-5}{2-3i}$.

UNIT II COMPLEX NUMBER AND TRIGONOMETRY

2.1 DEMOIVRE'S THEOREM

Statement:

i) If n is an integer, $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$

ii) If n is rational, then $(\cos n\theta + i\sin n\theta)$ is one of the values of $(\cos\theta + i\sin\theta)^n$

Note: i) If n is a (+) integer, then $(\cos\theta + i\sin\theta)^{-n} = (\cos(-n\theta) + i\sin(-n\theta))$

$= \cos n\theta - i\sin n\theta$

ii)If n is a (+) rational ,then $\cos n\theta - i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^{-n}$

Examples:

1) $(\cos\theta + i\sin\theta)^{-2} = \cos 2\theta - i\sin 2\theta$
2) $(\cos\theta + i\sin\theta)^{5} = \cos 5\theta + i\sin 5\theta$
3) $\frac{1}{(\cos\theta + i\sin\theta)^4} = (\cos\theta + i\sin\theta)^{-4} = \cos 4\theta - i\sin 4\theta$
4) $\frac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$
5) $\frac{1}{\cos 3\theta + i\sin 3\theta} = \cos 3\theta - i\sin 3\theta$
6) $\frac{1}{i} = \frac{i}{i.i} = \frac{i}{i^2} = -i$; $\therefore \frac{1}{i} = -i$

Note:

If $x = \cos\alpha + i\sin\alpha$ and $y = \cos\beta + i\sin\beta$, then

(i) $xy = \cos(\alpha + \beta) + i\sin(\alpha + \beta)$ and hence
$\frac{1}{xy} = \cos(\alpha + \beta) - i\sin(\alpha + \beta)$

(ii) $\frac{x}{y} = \cos(\alpha - \beta) + i\sin(\alpha - \beta)$ an hence
$\frac{y}{x} = \cos(\alpha - \beta) - i\sin(\alpha - \beta)$

WORKED EXAMPLES:

PART -A

1)Simplify: $\frac{\cos 3\theta + i\sin 3\theta}{\cos 2\theta - i\sin 2\theta}$

Solution: $\frac{\cos 3\theta + i\sin 3\theta}{\cos 2\theta - i\sin 2\theta} = \frac{(\cos\theta + i\sin\theta)^3}{(\cos\theta + i\sin\theta)^{-2}}$

$= (\cos\theta + i\sin\theta)^3 (\cos\theta + i\sin\theta)^2$

$= (\cos\theta + i\sin\theta)^{3+2}$

$= (\cos\theta + i\sin\theta)^5$

$= \cos 5\theta + i\sin 5\theta$

2) If x = cosθ + isinθ, what is the value of $x^m + \frac{1}{x^m}$

Solution:

Given x = cosθ + isinθ

X^m = (cosθ + isinθ)^m

= cosmθ + isinmθ

$\frac{1}{x^m}$ = (cos θ + isin θ)$^{-m}$

= cosmθ – isinmθ

$x^m + \frac{1}{x^m}$ = cosmθ + isinmθ + cosmθ – isinmθ

= 2 cosmθ

3) If x = cosθ + isinθ, what is the value of $x^5 - \frac{1}{x^5}$

Solution:

Given x = cosθ + isinθ

x^5 = (cosθ + isinθ)^5

= cos5θ + isin5θ

$\frac{1}{x^5}$ = (cos θ + isin θ)$^{-5}$

= cos5θ – isin5θ

$x^5 - \frac{1}{x^5}$ = cos5θ + isin5θ -(cos5θ – isin5θ)

= cos5θ + isin5θ - cos5θ + isin5θ

= 2isin5θ

4) If x= cosθ_1 + isinθ_1 and y = cosθ_2 + isinθ_2 then, find $x^m y^n$

Solution:

Given: x= cosθ_1 + isinθ_1 and y = cosθ_2 + isinθ_2

x^m = cosmθ_1 + isinmθ_1

y^n = cosnθ_2 + isinnθ_2

$x^m y^n$ = (cosmθ_1 + isinmθ_1)(cosnθ_2 + isinnθ_2)

= cos(m$\theta_1 + n\theta_2$) + isin(m$\theta_1 + n\theta_2$)

5) If x= $\cos\theta_1$ + i$\sin\theta_1$ and y = $\cos\theta_2$ + i$\sin\theta_2$ then, find $\frac{x^m}{y^n}$

Solution:

Given: x= $\cos\theta_1$ + i$\sin\theta_1$ and y = $\cos\theta_2$ + i$\sin\theta_2$

x^m = $\cos m\theta_1$ + i$\sin m\theta_1$ and y^n = $\cos n\theta_2$ + i$\sin n\theta_2$

$\frac{x^m}{y^n} = \frac{cosm\theta_1 + isinm\theta_1}{cosn\theta_2 + isinn\theta_2}$

= $\cos(m\theta_1 - n\theta_2)$ + i$\sin(m\theta_1 - n\theta_2)$

PART- B

1) If a = ($\cos 2\alpha$ + i$\sin 2\alpha$) ; b = ($\cos 2\beta$ + i$\sin 2\beta$) then, find $\sqrt{ab}$

Solution:

Given a = ($\cos 2\alpha$ + i$\sin 2\alpha$) ; b = ($\cos 2\beta$ + i$\sin 2\beta$)

$\sqrt{ab} = (ab)^{\frac{1}{2}} = [(\cos 2\alpha + i\sin 2\alpha)(\cos 2\beta + i\sin 2\beta)]^{\frac{1}{2}}$

$= [\cos(2\alpha + 2\beta) + i\sin(2\alpha + 2\beta)]^{\frac{1}{2}}$

$= [\cos 2(\alpha + \beta) + i\sin 2(\alpha + \beta)]^{\frac{1}{2}}$

$= cos2(\frac{\alpha+\beta}{2}) + isin2(\frac{\alpha+\beta}{2})$

$= \cos(\alpha + \beta) + i\sin(\alpha + \beta)$

2) Show that $\left(\frac{cosx+isinx}{sinx-icosx}\right)^4 = 1$

Solution:

Consider $\frac{cosx+isinx}{\sin x - i\cos x} = \frac{i(cosx+isinx)}{i(\sin x - i\cos x)}$

$= \frac{i(cosx+isinx)}{\cos x + i\sin x}$

= i

$\left(\frac{cosx+isinx}{sinx-icosx}\right)^4 = i^4$

$= i^2 . i^2$

= (-1) (-1)

=1

3) Prove that $\left(\frac{1+cosx+isinx}{1+cosx-isinx}\right)^n = \cos nx + i\sin nx$

Solution:

Let z = cos x + isin$x$

$\frac{1}{z}$ = cos x – isin$x$

$$\left(\frac{1+cosx+isinx}{1+cosx-isinx}\right)^n = \left(\frac{1+z}{1+\frac{1}{z}}\right)^n$$

$$= \left(\frac{1+z}{\frac{z+1}{z}}\right)^n$$

$$= \left(\frac{(1+z)z}{(z+1)}\right)^n$$

$$= z^n$$

$$= (\cos x + isin\ x)^n$$

= cos nx + isin nx

4) Simplify: (cos $\frac{\pi}{3}$ +isin $\frac{\pi}{3}$) (cos$\frac{\pi}{6}$ + isin$\frac{\pi}{6}$)

Solution:

Let z_1 = cos $\frac{\pi}{3}$ +isin $\frac{\pi}{3}$ and z_2 = cos$\frac{\pi}{6}$ + isin$\frac{\pi}{6}$

z_1z_2 = (cos $\frac{\pi}{3}$ +isin $\frac{\pi}{3}$) (cos$\frac{\pi}{6}$ + isin$\frac{\pi}{6}$)

= cos($\frac{\pi}{3}+\frac{\pi}{6}$) + isin($\frac{\pi}{3}$ + $\frac{\pi}{6}$)

= cos ($\frac{2\pi+\pi}{6}$) + isin($\frac{2\pi+\pi}{6}$)

= cos ($\frac{3\pi}{6}$) +isin($\frac{3\pi}{6}$)

= cos$\frac{\pi}{2}$ + isin$\frac{\pi}{2}$

= 0 +i

= i

PART - C

1) Simplify: $\frac{(cos3\theta+isin3\theta)(cos4\theta-isin4\theta)}{(os2\theta-isin2\theta)^5}$, when $\theta = \frac{\pi}{9}$

Solution: $\frac{(cos3\theta+isin3\theta)(cos4\theta-isin4\theta)}{(os2\theta-isin2\theta)^5} = \frac{(cos\theta+isin\theta)^3(cos\theta+isin\theta)^{-4}}{(os\theta+isin\theta)^{-10}}$

$= (\cos\theta + i\sin\theta)^{3-4+10}$

$= (\cos\theta + i\sin\theta)^{9}$

$= \cos 9\theta + i\sin 9\theta$

Let $\theta = \frac{\pi}{9}$, then

$\frac{(cos3\theta+isin3\theta)(cos4\theta-isin4\theta)}{(os2\theta-isin2\theta)^5} = \cos 9\frac{\pi}{9} + i\sin 9\frac{\pi}{9}$

$= \cos\pi + i\sin\pi$

$= -1 + i0$

$= -1$

2) Find the value of $\frac{(cos3\theta+isin3\theta)^4(cos4\theta-isin4\theta)^5}{(os4\theta+isin4\theta)^3\ (cos5\theta+isin5\theta)^{-4}}$

Solution:

$\frac{(cos3\theta+isin3\theta)^4(cos4\theta-isin4\theta)^5}{(os4\theta+isin4\theta)^3\ (cos5\theta+isin5\theta)^{-4}} = \frac{(cos\theta+isin\theta)^{3x4}(cos\theta+isin\theta)^{-4x5}}{(os\theta+isin\theta)^{4x3}\ (cos\theta+isin\theta)^{5x-4}}$

$= \frac{(cos\theta+isin\theta)^{12}(cos\theta+isin\theta)^{-20}}{(os\theta+isin\theta)^{12}\ (cos\theta+isin\theta)^{-20}}$

$= (\cos\theta + i\sin\theta)^{12-20-12+20}$

$= (\cos\theta + i\sin\theta)^{0}$

$= 1$

3) If a = cosx+isinx and b = cosy +isiny, then prove that

i)cos (x+y) = $\frac{1}{2}\left(ab + \frac{1}{ab}\right)$ ii)sin (x+y) = $\frac{1}{2i}\left(ab - \frac{1}{ab}\right)$

Solution:

Given: a = cos x +isin x and b = cosy +isiny

$\therefore$ ab = $\cos(x+y) + i\sin(x+y)$

$\frac{1}{ab} = \cos(x+y) - i\sin(x+y)$

i. To Prove cos (x+y) = $\frac{1}{2}\left(ab + \frac{1}{ab}\right)$

$\frac{1}{2}\left(ab + \frac{1}{ab}\right) = \frac{1}{2}[\cos(x+y) + i\sin(x+y) + \cos(x+y) - i\sin(x+y)]$

$= \frac{1}{2}[2\cos(x+y)]$

$= \cos(x+y)$

ii)To prove sin (x+y) = $\frac{1}{2i}\left(ab - \frac{1}{ab}\right)$

$\frac{1}{2i}\left(ab - \frac{1}{ab}\right)$ = $\frac{1}{2i}$ [cos($x + y$) + isin(x + y) -(cos($x + y$) - isin($x + y$))]

= $\frac{1}{2i}$ [cos($x + y$) + isin(x + y) -cos($x + y$) + isin($x + y$)]

= $\frac{1}{2i}$ [2isin(x+y)]

= sin(x+y)

4) If a = cos x+isin x and b = cos y +isin y, then prove that

i)cos (x-y) = $\frac{1}{2}\left(\frac{a}{b} + \frac{b}{a}\right)$ ii) sin (x-y)= $\frac{1}{2i}\left(\frac{a}{b} - \frac{b}{a}\right)$

Solution:

Given: a = cosx +isinx and b = cosy +isiny

$\frac{a}{b}$ = cos($x - y$) + isin(x - y)

$\frac{b}{a}$ = cos($x - y$) - isin(x - y)

i)To prove that cos (x-y) = $\frac{1}{2}\left(\frac{a}{b} + \frac{b}{a}\right)$

$\frac{1}{2}\left(\frac{a}{b} + \frac{b}{a}\right)$ = $\frac{1}{2}$ [cos($x - y$) + isin(x - y) + cos($x - y$) - isin(x - y)]

= $\frac{1}{2}$ [cos($x - y$) + cos($x - y$)]

= $\frac{1}{2}$ [2cos($x - y$)]

= cos(x – y)

ii)To prove that sin (x-y) = $\frac{1}{2i}\left(\frac{a}{b} - \frac{b}{a}\right)$

$\frac{1}{2i}\left(\frac{a}{b} - \frac{b}{a}\right)$ = $\frac{1}{2i}$ [cos($x - y$) + isin(x - y) -(cos($x - y$) - isin(x - y))]

= $\frac{1}{2i}$ [cos($x - y$) + isin(x - y) -cos($x - y$) + isin(x - y))]

= $\frac{1}{2i}$ 2isin(x - y)

= sin (x – y)

5) If a = cos α+isinα and b = cos β +isinβ then,

a) Prove that $\frac{a^2 - b^2}{2ab}$ = isin$(\alpha - \beta)$

b) find the value of i) $\sqrt[3]{ab} + \frac{1}{\sqrt[3]{ab}}$ ii) $\sqrt[3]{ab} - \frac{1}{\sqrt[3]{ab}}$

Solution:

Given: a = cos α +isinα and b = cos β +isin β

a) To prove $\frac{a^2-b^2}{2ab}$ = isin$(\alpha-\beta)$

L.H.S = $\frac{a^2-b^2}{2ab}$

$= \frac{1}{2}\left(\frac{a^2}{ab} - \frac{b^2}{ab}\right)$

$= \frac{1}{2}\left(\frac{a}{b} - \frac{b}{a}\right)$ ----------→(1)

$\frac{a}{b} = \frac{cos\alpha+isin\alpha}{cos\beta+isin\beta}$

$\frac{a}{b}$ = cos$(\alpha-\beta)$ + isin$(\alpha$ - $\beta)$

$\frac{b}{a}$ = cos$(\alpha-\beta)$ – isin$(\alpha$ - $\beta)$

From equation (1)

$\frac{a^2-b^2}{2ab} = \frac{1}{2}\left(\frac{a}{b} - \frac{b}{a}\right)$

$= \frac{1}{2}$ [cos$(\alpha-\beta)$ + isin$(\alpha-\beta)$ - (cos$(\alpha-\beta)$ - isin $(\alpha-\beta)$)]

$= \frac{1}{2}$ [cos$(\alpha-\beta)$ + isin$(\alpha-\beta)$ - cos$(\alpha-\beta)$ + isin $(\alpha-\beta)$]

$= \frac{1}{2}$ 2isin $(\alpha-\beta)$

= isin $(\alpha-\beta)$

b) i) To find the value $\sqrt[3]{ab} + \frac{1}{\sqrt[3]{ab}}$

ab = (cos α + isinα)(cosβ + isinβ)

= cos$(\alpha+\beta)$ + isin$(\alpha$ + $\beta)$

$\sqrt[3]{ab}$ = [(cosα + isinα)(cosβ + isinβ)]$^{\frac{1}{3}}$

$= cos(\frac{\alpha+\beta}{3}) + isin(\frac{\alpha+\beta}{3})$

$\frac{1}{\sqrt[3]{ab}} = cos\left(\frac{\alpha+\beta}{3}\right) - isin(\frac{\alpha+\beta}{3})$

$\sqrt[3]{ab} + \frac{1}{\sqrt[3]{ab}} = cos(\frac{\alpha+\beta}{3}) + isin(\frac{\alpha+\beta}{3}) + cos\left(\frac{\alpha+\beta}{3}\right) - isin(\frac{\alpha+\beta}{3})$

$= 2cos(\frac{\alpha+\beta}{3})$

ii) $\sqrt[3]{ab} - \frac{1}{\sqrt[3]{ab}} = cos(\frac{\alpha+\beta}{3}) + isin(\frac{\alpha+\beta}{3})$ - ($cos\left(\frac{\alpha+\beta}{3}\right) - isin(\frac{\alpha+\beta}{3})$)

$= cos(\frac{\alpha+\beta}{3}) + isin(\frac{\alpha+\beta}{3})$ - $cos(\frac{\alpha+\beta}{3}) + isin(\frac{\alpha+\beta}{3})$

$= 2isin(\frac{\alpha+\beta}{3})$

6) If a = cos x+isin x ; b = cos y +isin y and c = cos z +isin z, then prove that

i) $\left(\frac{ab}{c} + \frac{c}{ab}\right)$ = 2 cos (x+y-z) ii) $\left(\frac{ab}{c} - \frac{c}{ab}\right)$ = 2isin (x + y –z)

Solution:

Given: a = cos x +isin x ; b = cos y +isin y and c = cos z +isin z

ab = (cos x+isin x)(cos y+isin y)

= cos (x+y)+ isin(x+y)

i)To prove $\left(\frac{ab}{c} + \frac{c}{ab}\right)$ = 2 cos (x+y-z)

$\frac{ab}{c} = \frac{\cos(x+y)+isin(x+y)}{cosz+isinz}$

= cos(x+y-z)+isin(x+y-z)

$\frac{ab}{c} + \frac{c}{ab}$ = cos(x+y-z)+isin(x+y-z) + cos(x+y-z) - isin(x+y-z)

= 2 cos(x+y-z)

ii)To prove $\left(\frac{ab}{c} - \frac{c}{ab}\right)$ = 2i sin (x+y-z)

$\frac{ab}{c} - \frac{c}{ab}$ = cos(x+y-z)+isin(x+y-z) –(cos(x+y-z) - isin(x+y-z))

= cos(x+y-z)+isin(x+y-z) - cos(x+y-z)+isin(x+y-z)

= 2 isin(x+y-z)

Exercise

PART-A

1) Simplify: $(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$
2) Simplify: $\frac{cos5\theta+\text{isin}5\theta}{\cos2\theta-\text{isin}2\theta}$
3) Simplify: $\frac{cos7\theta-\text{isin}7\theta}{\cos3\theta-\text{isin}3\theta}$
4) Simplify: $\frac{cos2\theta-\text{isin}2\theta}{\cos4\theta+\text{isin}4\theta}$
5) Simplify: $\frac{cos6\theta+\text{isin}6\theta}{\cos8\theta+\text{isin}8\theta}$

PART-C

1) Prove that $\left(\frac{1+cosx+isinx}{1+cosx-isinx}\right)^4$ = cos4θ + isin4θ
2) Simplify: $\frac{(cosx-isinx)^3(cos3x+isin3x)^5}{(cos2x-isin2x)^5(cos5x+isin5x)^7}$ when $x = \frac{2\pi}{13}$
3) Find the value of $\frac{(cos2\theta+isin2\theta)^5(cos\theta-isin\theta)^3}{(\text{os}3\theta-\text{isin}3\theta)^2\ (\text{cos}5\theta+\text{isin}5\theta)^4}$
4) Find the value of $\frac{(cos5\theta-isin5\theta)^2(cos7\theta+isin7\theta)^{-3}}{(\text{os}4\theta-\text{isin}4\theta)^9\ (\text{cos}\theta-\text{isin}\theta)^{-5}}$
5) Find the value of $\frac{(cos2x-isin2x)^3}{(\text{os}3\text{x}-\text{isin}3\text{x})^{-5}\ (\text{cos}2\text{x}+\text{isin}2\text{x})^{-2}}$
6) Show that $\frac{(cos2\theta+isin2\theta)^2(cos3\theta-isin3\theta)^{-3}}{(\text{os}4\theta+\text{isin}4\theta)^{-6}\ (\text{cos}\theta+\text{isin}\theta)^8}$ = cos29θ + isin29θ
7) Using Demoivre's theorem, show that $\frac{(cos\theta+isin\theta)^4(cos3\theta+isin3\theta)^2}{(\text{os}2\theta-\text{isin}2\theta)^3\ (\text{cos}4\theta+\text{isin}4\theta)^4}$ = 1
8) If a = cosx+isinx and b = cosy +isiny then, find the value of
   i) $\sqrt[4]{ab}$ - $\frac{1}{\sqrt[4]{ab}}$ ii) $\sqrt[4]{ab}$ + $\frac{1}{\sqrt[4]{ab}}$
9) If a = cosx+isinx and b = cosy +isiny then, find the value of

   i) $\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)$ ii) $\left(\frac{a^2}{b^2}-\frac{b^2}{a^2}\right)$

10) If a = cosx+isinx ; b = cosy +isiny and c = cosz +isinz then, find the value of

   i) $\left(abc+\frac{1}{abc}\right)$ ii) $\left(abc-\frac{1}{abc}\right)$

2.2 ROOTS OF COMPLEX NUMBER

Let the complex number be z= r (cosθ +isinθ)

Then, ω is called an nth roots of a complex number z if ω^n = z and we write $\omega = z^{\frac{1}{n}}$

Method to find n distinct roots of nth roots of a complex number

Let Z = r (cosθ +isinθ)

= r [cos($2k\pi+\theta) + isin(2k\pi+\theta)$]

$z^{\frac{1}{n}} = r^{\frac{1}{n}}[\text{cos}(2k\pi+\theta)+isin(2k\pi+\theta)]^{\frac{1}{n}}$

= $r^{\frac{1}{n}}\left[\text{cos}\left(\frac{2k\pi+\theta}{n}\right)+isin(\frac{2k\pi+\theta}{n})\right]$ where k = 0, 1, 2, 3, ……(n-1)

By giving the values 0, 1, 2, 3, ……(n-1) to k ,we get all the n distinct roots

Note:

1. The number of the nth roots of a non – zero complex number is n
2. If ω is the nth roots of unity, then

i)$\omega^n = 1$ ii) $1 + \omega + \omega^2 + \omega^3 + \ldots + \omega^{n-1} = 0$ [ieThe sum of the roots is zero]

3. If ω is the cube roots of unity, then

   i)$\omega^3 = 1$ ii)$1 + \omega + \omega^2 = 0$

4. Cos0 + isin0 = 1
5. $\cos\pi + i\sin\pi$ = -1

<u>Working rule to find the nth roots of a complex number</u>

Step 1 : Express the given complex number in modulus and amplitude form (Polar form)

Step 2: Add 2kπ to the argument.

Step3: Apply Demoivre's theorem.

Step4: Put k = 0, 1, 2,3,4,……(n-1), so as to obtain n distinct roots.

<u>Worked Examples:</u>

<u>PART A</u>

1) If ω is the cube roots of unity, what is the value of $1 + \omega + \omega^2$?

   Solution:

   $1 + \omega + \omega^2 = 0$

2) If ω is the cube roots of unity, what is the value of ω^3 ?

   Solution:

   $\omega^3 = 1$

3) Simplify: $(1 + \omega)(1 + \omega^2)$

   Solution:

   $(1 + \omega)(1 + \omega^2) = 1 + \omega + \omega^2 + \omega^3$

   $= 0 + 1$ $[\, since\ 1 + \omega + \omega^2 = 0]$

   $= 1$

4) If ω is the cube roots of unity, what is the value of $1 + \omega^2 + \omega^4$?

   Solution:

   $1 + \omega^2 + \omega^4 = (1 + \omega^2) + \omega^3.\omega$

   $= -\omega + 1.\omega$ [since $1 + \omega + \omega^2 = 0$

   $= -\omega + \omega$ $1 + \omega^2 = -\omega$ and $\omega^3 = 1$]

   $= 0$

5) If ω is the cube roots of unity, find the value of $\omega^4 + \omega^5 + \omega^6$?

Solution:

$\omega^4 + \omega^5 + \omega^6 = \omega^3.\omega + \omega^3.\omega^2 + \omega^3.\omega^3$

$= 1.\omega + 1.\omega^2 + 1.1$ [since $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$]

$= \omega + \omega^2 + 1$

$= 0$

6) If ω is the cube roots of unity, find the value of $3\omega^{15} - 5$?

Solution:

$3\omega^{15} - 5 = 3(\omega^3)^5 - 5$

$= 3(1)^5 - 5$ [since $\omega^3 = 1$]

$= 3 - 5$

$= -2$

PART- B

1) Find the cube roots of unity (or) Solve $x^3 - 1 = 0$ (or) Find all values of $(1)^{\frac{1}{3}}$

Solution:

$x^3 - 1 = 0$

$x = 1^{\frac{1}{3}}$

$= (cos0 + isin0)^{\frac{1}{3}}$

$= [\cos(2k\pi + 0) + isin(2k\pi + 0)]^{\frac{1}{3}}$ where k = 0,1,2

$x = \cos\left(\frac{2k\pi}{3}\right) + isin(\frac{2k\pi}{3})$ where k = 0, 1, 2 ----------→ (1)

Put k = 0, 1, 2 in (1)

When k = 0

$x = \cos\left(\frac{2(0)\pi}{3}\right) + isin(\frac{2(0)\pi}{3})$

$= \cos 0 + i\sin 0$

$= 1$

When k = 1

$x = \cos\left(\frac{2(1)\pi}{3}\right) + isin(\frac{2(1)\pi}{3})$

$= \cos\left(\frac{2\pi}{3}\right) + isin(\frac{2\pi}{3})$

$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

When k = 2

$x = \cos\left(\frac{2(2)\pi}{3}\right) + isin(\frac{2(2)\pi}{3})$

$= \cos\left(\frac{4\pi}{3}\right) + isin(\frac{4\pi}{3})$

$= -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

2) Find the cube roots of -1 (or) Solve $x^3 + 1 = 0$ (or) Solve $x = (-1)^{\frac{1}{3}}$

Solution:

$x^3 + 1 = 0$

$x^3 = -1$

$x = (-1)^{\frac{1}{3}}$

$= (cos\pi + isin\pi)^{\frac{1}{3}}$

$= [\cos(2k\pi + \pi) + isin(2k\pi + \pi)]^{\frac{1}{3}}$ where k = 0,1,2

$x = \cos\left(\frac{2k\pi+\pi}{3}\right) + isin(\frac{2k\pi+\pi}{3})$ where k = 0, 1, 2 ----------→ (1)

Put k = 0, 1, 2 in (1)

When k = 0

$x = \cos\left(\frac{2(0)\pi+\pi}{3}\right) + isin(\frac{2(0)\pi+\pi}{3})$

$= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

When k = 1

$x = \cos\left(\frac{2(1)\pi+\pi}{3}\right) + isin(\frac{2(1)\pi+\pi}{3})$

$= \cos\left(\frac{3\pi}{3}\right) + isin(\frac{3\pi}{3})$

$= \cos\pi + isin\pi$

When k = 2

$x = \cos\left(\frac{2(2)\pi+\pi}{3}\right) + isin(\frac{2(2)\pi+\pi}{3})$

$= \cos\left(\frac{5\pi}{3}\right) + isin(\frac{5\pi}{3})$

PART- C

1) If 1, ω, ω^2 are the cube roots of unity, prove that $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 = 32$

Solution:

$(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 = (-2\omega)^5 + (-2\omega^2)^5$

$= (-2)^5\ \omega^5 + \ (-2)^5\ \omega^{10}$ [since 1, ω, ω^2 are the cube roots of unity]

$= (-2)^5(\omega^5 + \ \omega^{10}) = (-2)^5(\omega^3\omega^2 + \omega^9\omega)$

$= -2^5\ (\ 1.\omega^2 + \ 1.\omega\ )$

$= -2^5\ (\ \omega^2 + \ \omega\ )$

$= -2^5$ (-1) [since $1 + \ \omega + \ \omega^2 = 0$]

= 32

2) Find the sixth roots of unity (or) Solve $x^6 - 1 = 0$

Solution:

$x^6 - 1 = 0$

$x = \ 1^{\frac{1}{6}}$

$= (cos0 + isin0)^{\frac{1}{6}}$

$= [\cos(2k\pi + 0) + \ isin(2k\pi + 0\ )]^{\frac{1}{6}}$ where k = 0,1,2,3,4,5

$x = \cos\left(\frac{2k\pi}{6}\right) + \ isin(\frac{2k\pi}{6})$ where k = 0, 1, 2 ,3,4,5----------→ (1)

Put k = 0, 1, 2,3,4,5 in (1)

When k = 0

$x = \cos\left(\frac{2(0)\pi}{6}\right) + \ isin(\frac{2(0)\pi}{6})$

= cos 0 + isin 0 = 1

When k = 1

$x = \cos\left(\frac{2(1)\pi}{6}\right) + \ isin(\frac{2(1)\pi}{6})$

$= \ \cos\left(\frac{2\pi}{6}\right) + \ isin(\frac{2\pi}{6}) = = \ \cos\left(\frac{\pi}{3}\right) + \ isin(\frac{\pi}{3})$

When k = 2

$x = \cos\left(\frac{2(2)\pi}{6}\right) + \ isin(\frac{2(2)\pi}{6})$

$= \cos\left(\frac{4\pi}{6}\right) + \ isin(\frac{4\pi}{6}) = \cos\left(\frac{2\pi}{3}\right) + \ isin(\frac{2\pi}{3})$

When k = 3

$x = \cos\left(\frac{2(3)\pi}{6}\right) + \ isin(\frac{2(3)\pi}{6})$

$= \cos\left(\frac{6\pi}{6}\right) + isin(\frac{6\pi}{6}) = \cos\pi + isin\,\pi$

When k = 4

$x = \cos\left(\frac{2(4)\pi}{6}\right) + isin(\frac{2(4)\pi}{6})$

$= \cos\left(\frac{8\pi}{6}\right) + isin(\frac{8\pi}{6}) = \cos\left(\frac{4\pi}{3}\right) + isin(\frac{4\pi}{3})$

When k = 5

$x = \cos\left(\frac{2(5)\pi}{6}\right) + isin(\frac{2(5)\pi}{6})$

$= \cos\left(\frac{10\pi}{6}\right) + isin(\frac{10\pi}{6}) = \cos\left(\frac{5\pi}{3}\right) + isin(\frac{5\pi}{3})$

3) Find the seventh roots of -1 (or) Solve $x^7 + 1 = 0$ (or) Find all value of $(-1)^{\frac{1}{7}}$

Solution:

$x^7 + 1 = 0$

$x^7 = -1$

$x = (-1)^{\frac{1}{7}}$

$= (cos\pi + isin\pi)^{\frac{1}{7}}$

$= [\cos(2k\pi + \pi) + isin(2k\pi + \pi)]^{\frac{1}{7}}$ where k = 0,1,2,3,4,5,6

$x = \cos\left(\frac{2k\pi+\pi}{7}\right) + isin(\frac{2k\pi+\pi}{7})$ where k = 0, 1, 2 ,3,4,5,6 → (1)

Put k = 0, 1, 2,3,4,5,6 in (1)

When k = 0

$x = \cos\left(\frac{2(0)\pi+\pi}{7}\right) + isin(\frac{2(0)\pi+\pi}{7})$

$= \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$

When k = 1

$x = \cos\left(\frac{2(1)\pi+\pi}{7}\right) + isin(\frac{2(1)\pi+\pi}{7})$

$= \cos\left(\frac{3\pi}{7}\right) + isin(\frac{3\pi}{7})$

$= \cos\frac{3\pi}{7} + isin\frac{3\pi}{7}$

When k = 2

$$x = \cos\left(\frac{2(2)\pi+\pi}{7}\right) + isin(\frac{2(2)\pi+\pi}{7})$$

$$= \cos\left(\frac{5\pi}{7}\right) + isin(\frac{5\pi}{7})$$

When k = 3

$$x = \cos\left(\frac{2(3)\pi+\pi}{7}\right) + isin(\frac{2(3)\pi+\pi}{7})$$

$$= \cos\left(\frac{7\pi}{7}\right) + isin(\frac{7\pi}{7}) = = \cos\pi + isin\,\pi$$

When k = 4

$$x = \cos\left(\frac{2(4)\pi+\pi}{7}\right) + isin(\frac{2(4)\pi+\pi}{7})$$

$$= \cos\left(\frac{9\pi}{7}\right) + isin(\frac{9\pi}{7})$$

When k = 5

$$x = \cos\left(\frac{2(5)\pi+\pi}{7}\right) + isin(\frac{2(5)\pi+\pi}{7})$$

$$= \cos\left(\frac{11\pi}{7}\right) + isin(\frac{11\pi}{7})$$

When k = 6

$$x = \cos\left(\frac{2(6)\pi+\pi}{7}\right) + isin(\frac{2(6)\pi+\pi}{7})$$

$$x = \cos\left(\frac{13\pi}{7}\right) + isin(\frac{13\pi}{7})$$

4)Find the fifth roots of unity (or) Solve $x^5 - 1 = 0$ (or) Find all value of $(1)^{\frac{1}{5}}$

Solution:

$$x^5 - 1 = 0$$

$$x = 1^{\frac{1}{5}}$$

$$= (cos0 + isin0)^{\frac{1}{5}}$$

$$= [\cos(2k\pi + 0) + isin(2k\pi + 0)]^{\frac{1}{5}} \quad \text{where } k = 0,1,2,3,4$$

$$x = \cos\left(\frac{2k\pi}{5}\right) + isin(\frac{2k\pi}{5}) \text{ where } k = 0, 1, 2, 3, 4 \text{----------} \rightarrow (1)$$

Put k = 0, 1, 2,3,4 in (1)

When k = 0

$$x = \cos\left(\frac{2(0)\pi}{5}\right) + isin(\frac{2(0)\pi}{5})$$

$$= \cos 0 + i\sin 0 = 1$$

When k = 1

$$x = \cos\left(\frac{2(1)\pi}{5}\right) + isin(\frac{2(1)\pi}{5})$$

$$= \cos\left(\frac{2\pi}{5}\right) + isin(\frac{2\pi}{5})$$

When k = 2

$$x = \cos\left(\frac{2(2)\pi}{5}\right) + isin(\frac{2(2)\pi}{5})$$

$$= \cos\left(\frac{4\pi}{5}\right) + isin(\frac{4\pi}{5})$$

When k = 3

$$x = \cos\left(\frac{2(3)\pi}{5}\right) + isin(\frac{2(3)\pi}{5})$$

$$= \cos\left(\frac{6\pi}{5}\right) + isin(\frac{6\pi}{5})$$

When k = 4

$$x = \cos\left(\frac{2(4)\pi}{5}\right) + isin(\frac{2(4)\pi}{5})$$

$$= \cos\left(\frac{8\pi}{5}\right) + isin(\frac{8\pi}{5})$$

5) Solve: $x^5+x^3+x^2+1 = 0$

Solution:

$x^5+x^3+x^2+1 = 0$

$x^3(x^2+1)+(x^2+1)= 0$

$(x^3+1)(x^2+1) = 0$

$x^3+1 =0$ and $x^2 + 1= 0$

$x^3 + 1 = 0$

consider, $x^3 = -1$

$x = (-1)^{\frac{1}{3}}$

$= (cos\pi + isin\pi)^{\frac{1}{3}}$

$= [\cos(2k\pi + \pi) + isin(2k\pi + \pi)]^{\frac{1}{3}}$ where k = 0,1,2

$x = \cos\left(\frac{2k\pi+\pi}{3}\right) + isin(\frac{2k\pi+\pi}{3})$ where k = 0, 1, 2 ----------→ (1)

Put k = 0, 1, 2 in (1)

When k = 0

$x = \cos\left(\frac{2(0)\pi+\pi}{3}\right) + isin(\frac{2(0)\pi+\pi}{3})$

$= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

When k = 1

$x = \cos\left(\frac{2(1)\pi+\pi}{3}\right) + isin(\frac{2(1)\pi+\pi}{3})$

$= \cos\left(\frac{3\pi}{3}\right) + isin(\frac{3\pi}{3})$

$= \cos\pi + isin\pi$

When k = 2

$x = \cos\left(\frac{2(2)\pi+\pi}{3}\right) + isin(\frac{2(2)\pi+\pi}{3})$

$= \cos\left(\frac{5\pi}{3}\right) + isin(\frac{5\pi}{3})$

Consider, $x^2 + 1 = 0$

$x^2 = -1$

$x = (-1)^{\frac{1}{2}}$

$= (cos\pi + isin\pi)^{\frac{1}{2}}$

$= [\cos(2k\pi + \pi) + isin(2k\pi + \pi)]^{\frac{1}{2}}$ where k = 0,1

$x = \cos\left(\frac{2k\pi+\pi}{2}\right) + isin(\frac{2k\pi+\pi}{2})$ where k = 0, 1 → (1)

Put k = 0, 1 in (1)

When k = 0

$x = \cos\left(\frac{2(0)\pi+\pi}{2}\right) + isin(\frac{2(0)\pi+\pi}{2})$

$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

When k = 1

$x = \cos\left(\frac{2(1)\pi+\pi}{2}\right) + isin(\frac{2(1)\pi+\pi}{2})$

$= \cos\left(\frac{3\pi}{2}\right) + isin(\frac{3\pi}{2})$

6) Solve: $x^5+x^3-x^2-1 = 0$

Solution:

$x^5+x^3-x^2-1 = 0$

$x^3(x^2+1)-(x^2+1)= 0$

$(x^3-1)(x^2+1) = 0$

$x^3-1 =0$ and $x^2 + 1= 0$

$x^3 - 1 = 0$

$x^3 = 1$

$x = (1)^{\frac{1}{3}}$

$= (cos0 + isin0)^{\frac{1}{3}}$

$= [\cos(2k\pi + 0) + isin(2k\pi + 0)]^{\frac{1}{3}}$ where k = 0,1,2

$x = \cos\left(\frac{2k\pi}{3}\right) + isin(\frac{2k\pi}{3})$ where k = 0, 1, 2→ (1)

Put k = 0, 1, 2 in (1)

When k = 0

$x = \cos\left(\frac{2(0)\pi}{3}\right) + isin(\frac{2(0)\pi}{3})$

$= \cos 0 + i\sin 0$

When k = 1

$x = \cos\left(\frac{2(1)\pi}{3}\right) + isin(\frac{2(1)\pi}{3})$

$= \cos\left(\frac{2\pi}{3}\right) + isin(\frac{2\pi}{3})$

When k = 2

$x = \cos\left(\frac{2(2)\pi}{3}\right) + isin(\frac{2(2)\pi}{3})$

$= \cos\left(\frac{4\pi}{3}\right) + isin(\frac{4\pi}{3})$

$x^2 + 1 = 0$

$x^2 = -1$

$x = (-1)^{\frac{1}{2}}$

$= (cos\pi + isin\pi)^{\frac{1}{2}}$

$= [\cos(2k\pi + \pi) + isin(2k\pi + \pi)]^{\frac{1}{2}}$ where k = 0,1

$x = \cos\left(\frac{2k\pi+\pi}{2}\right) + isin(\frac{2k\pi+\pi}{2})$ where k = 0, 1 → (1)

Put k = 0, 1 in (1)

When k = 0

$x = \cos\left(\frac{2(0)\pi+\pi}{2}\right) + isin(\frac{2(0)\pi+\pi}{2})$

$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

When k = 1

$x = \cos\left(\frac{2(1)\pi+\pi}{2}\right) + isin(\frac{2(1)\pi+\pi}{2})$

$= \cos\left(\frac{3\pi}{2}\right) + isin(\frac{3\pi}{2})$

EXERCISE

PART-A

1) If ω is the cube roots of unity, what is the value of

   i) $\omega^9 + \omega^6 + \omega^3$ ii) $3 - \omega^{15} - \omega^3$

2) If ω is the cube roots of unity, what is the value of $(1 + \omega)$?

3) If ω is the cube roots of unity, what is the value of $\omega + \omega^2$?

4) If ω is the nth roots of unity, what is the value of $1 + \omega + \omega^2 + \ldots\ldots\ldots\ldots \omega^{n-1}$?

5) If ω is the cube roots of unity, what is the value of $\omega(\omega + 1)$?

PART-B

1) Solve : $x^2 - 1 = 0$

2) Solve : $x^2 + 1 = 0$

PART-C

1) Solve : $x^7 - 1 = 0$
2) Solve : $x^6 + 1 = 0$
3) Solve: $x^4 - 1 = 0$
4) Solve: $x^4 + 1 = 0$
5) Solve: $x^7 + x^4 + x^3 + 1 = 0$
6) Solve: $x^6 + x^4 + x^2 + 1 = 0$
7) Solve: $x^8 + x^5 + x^3 + 1 = 0$
8) Solve: $x^8 + x^5 - x^3 - 1 = 0$
9) Solve: $x^8 - x^5 + x^3 - 1 = 0$
10) Solve : $x^5 + 1 = 0$

2.3 COMPOUND ANGLES

Definition:

An angle which is made up of sum or difference of two or more angles is called a compound angle.

Example: A + B , A – B, A+ B + C , A – B + C , etc ……, are compound angles.

It is important to note that the relation f (x + y) = f (x) + f (y) is not true for all functions of a real variable. As an example , all the six trigonometric functions do not satisfy the above relation.

ie) $\sin(x+y) \neq \sin x + \sin y$ …… (1)

$\tan(x+y) \neq \tan x + \tan y$

This can be easily verified by putting $x = 60^\circ$ and $y = 30^\circ$ in (1)

Then $x + y = 90^\circ$

$\therefore \sin(x+y) = \sin(60^\circ + 30^\circ)$

$= \sin 90^\circ$

$\sin(x+y) = 1$ ……(2)

But, $\sin x + \sin y = \sin 60^\circ + \sin 30^\circ$

$\sin x + \sin y = \frac{\sqrt{3}}{2} + \frac{1}{2}$ ……(3)

From (2) and (3), $\sin(x+y) \neq \sin x + \sin y$

Also, sin (x+ y) ≠ sin x + y (Meaningless)

Expansions of sin(A ± B), cos (A ± B) and tan (A ± B)

The following are the relationship between the trigonometric functions of compound angles in terms of those of the separate angles :

<u>Sine addition formula:</u>

i)Sin(A+B) = SinA CosB +CosA SinB

ii)Sin(A-B) = SinA CosB – CosA SinB

<u>cosine addition formula:</u>

iii)Cos(A+B) = CosA CosB –SinASinB

iv)Cos(A-B) = CosA CosB + SinA SinB

<u>Tangent addition formula:</u>

v) tan(A+B) = $\frac{tanA+tanB}{1-tanA\ tanB}$

vi)tan(A-B)= $\frac{tanA-tanB}{1+tanAtanB}$

Note:

i)cot (A+B) = $\frac{1}{\tan(A+B)}$

= $\frac{1-tanAtanB}{tanA+tanB}$

ii)cot(A-B) = $\frac{1+tanAtanB}{tanA-tanB}$

iii)sin(A+B) + sin(A-B) = (sinAcosB + cosAsinB) + (sinAcosB - cosAsinB)

= sinAcosB + sinAcosB

= 2sinAcosB

Sin(A+B) + sin(A – B)= 2sinAcosB

v) sin(A+B) - sin(A-B) = (sinAcosB + cosAsinB) - (sinAcosB - cosAsinB)

=sinAcosB + cosAsinB - sinAcosB + cosAsinB

= cosAsinB + cosAsinB

= 2cosAsinB

Sin(A+B) - sin(A – B)= 2cosAsinB (or) Sin(A-B) –sin(A+B) = - 2cosAsinB

vi) cos(A+B) + cos(A-B) = (cosAcosB - sinAsinB) + (cosAcosB + sinAsinB)

= cosAcosB + cosAcosB

= 2cosAcosB

$\boxed{\cos(A+B) + \cos(A - B) = 2\cos A\cos B}$

vii) cos(A+B) - cos(A-B) = (cosAcosB - sinAsinB) - (cosAcosB + sinAsinB)

=cosAcosB - sinAsinB - cosAcosB - sinAsinB

= - sinAsinB - sinAsinB

= -2sinAsinB

$\boxed{\begin{array}{c}\cos(A+B) - \cos(A - B) = -2\sin A\sin B \\ \text{(or)} \\ \cos(A-B) - \cos(A+B) = 2\sin A\sin B\end{array}}$

WORKED EXAMPLES

PART - A

1) Without using tables, find the value of $\sin 38^\circ \cos 22^\circ + \cos 38^\circ \sin 22^\circ$

Solution:

Sin(A+B) = SinA CosB +CosA SinB

$\sin 38^\circ \cos 22^\circ + \cos 38^\circ \sin 22^\circ = \sin(38^\circ + 22^\circ)$

$= \sin 60^\circ$

$= \frac{\sqrt{3}}{2}$

2) Without using tables, find the value of $\frac{\tan 100^\circ - \tan 10^\circ}{1 + \tan 100^\circ \tan 10^\circ}$

Solution:

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$\frac{\tan 100^\circ - \tan 10^\circ}{1 + \tan 100^\circ \tan 10^\circ} = \tan(100^\circ - 10^\circ)$

$= \tan 90^\circ$

$= \infty$

3) Prove that $\sin(60^\circ - A)\cos(30^\circ + A) + \cos(60^\circ - A)\sin(30^\circ + A) = 1$

Solution:

Sin(A+B) = SinA CosB +CosA SinB

$\sin(60^\circ - A)\cos(30^\circ + A) + \cos(60^\circ - A)\sin(30^\circ + A) = \sin[(60^\circ - A) + (30^\circ + A)]$

$= \sin[60^\circ - A + 30^\circ + A]$

$= \sin 90^\circ$

$= 1$

4) Without using tables, find the value of $\sin 15^\circ$

Solution:

Sin(A-B) = SinA CosB - CosA SinB

$\sin 15^\circ = \sin(45^\circ - 30^\circ)$

$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$

$= \frac{1}{\sqrt{2}} \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \frac{1}{2}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

5) Without using tables, find the value of $\cos 75^\circ$

Solution:

cos(A+B) = cosA CosB - sinA SinB

$\cos 75^\circ = \cos(45^\circ + 30^\circ)$

$= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$

$= \frac{1}{\sqrt{2}} \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \frac{1}{2}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

6) Prove that $\frac{\sin(A+B)+\sin(A-B)}{\cos(A+B)+\cos(A-B)} = \tan A$

Proof:

$\frac{\sin(A+B)+\sin(A-B)}{\cos(A+B)+\cos(A-B)} = \frac{2 sinA cosB}{2 cosA cosB}$

$= \frac{\sin A}{cosA}$

$= \tan A$

PART- B

1) Prove that $\tan A + \tan B = \frac{\sin(A+B)}{cosA cosB}$

Proof:

$\frac{\sin(A+B)}{cosAcosB} = \frac{\text{sinAcosB+cosAsinB}}{cosAcosB}$ [since Sin(A+B) = SinA CosB +CosA SinB]

$= \frac{\sin AcosB}{cosAcosB} + \frac{\cos AsinB}{cosAcosB}$

$= \frac{\text{sinA}}{cosA} + \frac{\sin B}{cosB}$

= tanA + tanB

2)If tanx = $\frac{5}{6}$ and tany = $\frac{1}{11}$,then prove that tan(x+y) = 1

Solution:

tanx = $\frac{5}{6}$ and tany = $\frac{1}{11}$

We know that tan(A+B) = $\frac{tanA+tanB}{1-tanA\ tanB}$

tan(x+y) = $\frac{\frac{5}{6}+\frac{1}{11}}{1-\frac{5}{6}\cdot\frac{1}{11}}$

$= \frac{\frac{55+6}{66}}{\frac{66-5}{66}}$

$= \frac{61}{66} \times \frac{66}{61}$

= 1

tan(x+y) = 1

3) Without using tables, find the value of tan 15°

Solution:

tan(A-B) = $\frac{tanA-tanB}{1+tanA\ tanB}$

tan 15° = tan($45^\circ - 30^\circ$)

$= \frac{tan45^\circ - tan30^\circ}{1+tan45^\circ\ tan30^\circ}$

$= \frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}$

$= \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}}$

$= \frac{\sqrt{3}-1}{\sqrt{3}+1}$

Multiply both (Nr) and (Dr) by $\sqrt{3}$ - 1

$\tan 15^\circ = \frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{(\sqrt{3}-1)^2}{(\sqrt{3})^2 - 1}$

$= \frac{3+1-2\sqrt{3}}{3-1}$

$= \frac{4-2\sqrt{3}}{2}$

$= \frac{2(2-\sqrt{3})}{2}$

$= 2- \sqrt{3}$

PART - C

1)If A and B are acute angles such that sinA = $\frac{3}{5}$ and cos B = $\frac{12}{13}$, then find the values of i) sin(A –B) ii) prove that cos (A+B) = $\frac{33}{65}$

Solution:

Given: sinA = $\frac{3}{5}$ and cos B = $\frac{12}{13}$

cosA = $\sqrt{1 - sin^2 A}$

$= \sqrt{1 - \left(\frac{3}{5}\right)^2}$

$= \sqrt{1 - \frac{9}{25}}$

$= \sqrt{\frac{16}{25}}$

$= \frac{4}{5}$

sinB = $\sqrt{1 - cos^2 A}$

$= \sqrt{1 - \left(\frac{12}{13}\right)^2}$

$= \sqrt{1 - \frac{144}{169}}$

$= \sqrt{\frac{25}{169}}$

$= \frac{5}{13}$

Sin (A-B) = sinAcosB – cosAsinB

$= \frac{3}{5} \cdot \frac{12}{13} - \frac{4}{5} \cdot \frac{5}{13}$

$= \frac{36}{65} - \frac{20}{65}$

$= \frac{16}{65}$

ii)Cos(A+B) = cosAcosB – sinAsinB

$= \frac{4}{5} \cdot \frac{12}{13} - \frac{3}{5} \cdot \frac{5}{13}$

$= \frac{48}{65} - \frac{15}{65}$

$= \frac{33}{65}$

2) If A and B are acute angles such that sinA = $\frac{1}{\sqrt{10}}$ and sinB = $\frac{1}{\sqrt{5}}$, then prove that A + B = $\frac{\pi}{4}$

Solution:

Given: sinA = $\frac{1}{\sqrt{10}}$ and sinB = $\frac{1}{\sqrt{5}}$

cosB = $\sqrt{1 - sin^2 B}$

$= \sqrt{1 - \left(\frac{1}{\sqrt{5}}\right)^2}$

$= \sqrt{1 - \frac{1}{5}}$

$= \sqrt{\frac{4}{5}}$

$= \frac{2}{\sqrt{5}}$

cosA = $\sqrt{1 - sin^2 A}$

$= \sqrt{1 - \left(\frac{1}{\sqrt{10}}\right)^2}$

$= \sqrt{1 - \frac{1}{10}}$

$= \sqrt{\frac{9}{10}}$

$= \frac{3}{\sqrt{10}}$

Sin (A+B) = sinAcosB + cosAsinB

$= \frac{1}{\sqrt{10}} \cdot \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{10}} \cdot \frac{1}{\sqrt{5}}$

$= \frac{2}{\sqrt{10}\sqrt{5}} + \frac{3}{\sqrt{10}\sqrt{5}}$

$= \frac{5}{\sqrt{10}\sqrt{5}}$

$= \frac{\sqrt{5}\sqrt{5}}{\sqrt{10}\sqrt{5}}$

$= \frac{\sqrt{5}}{\sqrt{10}}$

$= \sqrt{\frac{5}{10}}$

Sin(A+ B)= $\frac{1}{\sqrt{2}}$

Sin(A+ B) = sin45°

A + B = $\frac{\pi}{4}$

3)If A + B = 45°,prove that (1 + tanA) (1+tanB) = 2 and hence deduce the value of tan$22\frac{1}{2}^\circ$

Solution:

Given: A + B = 45°

Taking tangent function on both sides

tan(A+ B) = tan45°

$\frac{tanA+tanB}{1-tanA\ tanB}$ = 1

tanA + tanB = 1 – tanA tanB

tanA + tanB + tanA tanB = 1…….(1)

(1+tanA)(1+tanB) = 1 + tanA + tanB + tanAtanB

= 1 + 1 by (1)

=2

Put A = $22\frac{1}{2}^\circ$, B = $22\frac{1}{2}^\circ$, then A + B = $22\frac{1}{2}^\circ + 22\frac{1}{2}^\circ = 45^\circ$

(1 + tan$22\frac{1}{2}^\circ$) (1 + tan $22\frac{1}{2}^\circ$) = 2

$\left(1 + tan22\frac{1}{2}^\circ\right)^2$ = 2

1 + tan $22\frac{1}{2}^\circ = \sqrt{2}$ (Since A and B are acute angle)

tan$22\frac{1}{2}^\circ$ = $\sqrt{2}$ -1

4) Prove that $\frac{\sin(A-B)}{sinAsinB} + \frac{\sin(B-C)}{sinBsinC} + \frac{\sin(C-A)}{sinCsinA} = 0$

Solution:

We know that Sin (A-B) = sinAcosB – cosAsinB

$\frac{\sin(A-B)}{sinAsinB} = \frac{\text{sinAcosB}-\text{cosAsinB}}{sinAsinB}$

$= \frac{\sin AcosB}{sinAsinB} - \frac{\cos AsinB}{sinAsinB}$

$= \frac{cosB}{sinB} - \frac{\cos A}{sinA}$

= cotB – cotA……..(1)

Similarly, we can prove that

$\frac{\sin(B-C)}{sinBsinC}$ = cotC – cotB ……(2)

$\frac{\sin(C-A)}{sinCsinA}$ = cotA – cotC……(3)

$\frac{\sin(A-B)}{sinAsinB} + \frac{\sin(B-C)}{sinBsinC} + \frac{\sin(C-A)}{sinCsinA}$ = cotB – cotA + cotC – cotB+ cotA – cotC [from (1) (2) and (3)]

= 0

$\sum \frac{\sin(A-B)}{sinAsinB} = 0$

<u>Exercise</u>

<u>PART-A</u>

1)Without using tables find the values for the following

i) $\sin 75^\circ$ ii) $\sin 105^\circ$ iii) $\cos 15^\circ$ iv) $\cos 105^\circ$ v) $\tan 75^\circ$ vi) $\tan 105^\circ$

2)without using tables find the values for the following

i) $\sin 75^\circ \cos 15^\circ - \cos 75^\circ \sin 15^\circ$ ii) $\cos 55^\circ \cos 25^\circ + \sin 55^\circ \sin 25^\circ$

iii) $\cos 50^\circ \cos 40^\circ - \cos 50^\circ \sin 40^\circ$ iv) $\cos 70^\circ \cos 10^\circ + \sin 70^\circ \sin 10^\circ$

v) $\sin 65^\circ \cos 25^\circ + \cos 65^\circ \sin 25^\circ$ vi) $\sin 22^\circ \cos 23^\circ + \cos 22^\circ \sin 23^\circ$

<u>PART-B</u>

1) Prove the following:

i) $\cos(60^\circ + A)\cos(30^\circ - A) + \sin(60^\circ + A)\sin(30^\circ + A) = 0$

ii) $\cos(A + 30^\circ)\cos(A - 30^\circ) + \sin(A + 30^\circ)\sin(A - 30^\circ) = \frac{1}{2}$

iii) $\sin(60^\circ - A)\cos(30^\circ + A) + \cos(60^\circ - A)\sin(30^\circ + A) = 1$

2) Find the value of $\frac{tan22^\circ + tan23^\circ}{1 - tan22^\circ\, tan23^\circ}$

3) Find the value of $\frac{tan65^\circ - tan20^\circ}{1+tan65^\circ\ tan20^\circ}$

4) Find the value of $\frac{tan18^\circ + tan12^\circ}{1-tan18^\circ\ tan12^\circ}$

5) Find the value of $\frac{tan85^\circ - tan25^\circ}{1+\ tan85^\circ\ tan25^\circ}$

6) Prove that $\frac{tan4A - \ tan3A}{1+tan4A\ tan3A}$ = tanA

7) Prove that $\frac{\sin(A-B)}{sinAsinB}$ = cotB - cotA

8) If tanA = $\frac{2}{11}$ and tanB = $\frac{7}{24}$, show that tan(A+B) = $\frac{1}{2}$

9) If tanx = $\frac{1}{2}$ and tany = $\frac{1}{3}$,then find the value of tan(x - y)

PART-C

1) If tanA = $\frac{18}{17}$ and tanB = $\frac{1}{35}$,show that A – B = 45°

2) If A and B are acute angles such that cosA = $\frac{1}{7}$ and cosB = $\frac{13}{14}$ then prove that A - B = $\frac{\pi}{3}$

3) If A and B are acute angles such that sinA = $\frac{4}{5}$ and sin B = $\frac{40}{41}$,find sin(A – B) and cos(A-B)

4) If A and B are acute angles such that sinA= $\frac{8}{17}$ and sin B = $\frac{5}{13}$ then prove that sin(A+ B) = $\frac{171}{221}$

5) Prove that $\frac{\sin(A-B)}{cosAcosB} + \frac{\sin(B-C)}{cosBcosC} + \frac{\sin(C-A)}{cosCcosA} = 0$

<u>Trigonometric functions of Multiple angle 2A and 3A in terms of trigonometric functions of angle A</u>

1) Sin 2A = 2 sinA cosA
2) Sin2A = $\frac{2tanA}{1+\ tan^2A}$
3) Cos2A = $cos^2A - \ sin^2A$
4) Cos2A = $2cos^2A - \ 1$
5) Cos2A = $1 - \ 2sin^2A$
6) Cos2A = $\frac{1-\ tan^2A}{1+\ tan^2A}$
7) tan2A = $\frac{2\ tanA}{1-\ tan^2A}$
8) sin3A = 3sinA – 4 sin^3A
9) cos3A = 4 $cos^3A - 3\ cosA$
10) tan3A = $\frac{3tanA - \ tan^3A}{1-3tan^2A}$

<u>Note:</u>

1) $sin^2A = \frac{1-cos2A}{2}$
2) $cos^2A = \frac{1+cos2A}{2}$
3) $sin^3A = \frac{3sinA\ -sin3A}{4}$

4) $cos^3A = \frac{3cosA + cos3A}{4}$

Sum and Product formulae:

This chapter deals with the transformation of product of two trigonometric functions into sum or difference of trigonometric functions and the transformation of sum or difference into product

Sum formulae: Used to express product of two trigonometric ratios into sum (or) difference

1) 2sinA cos B = sin(A + B) + sin (A – B)
2) 2cosA sinB = sin (A + B) – sin (A – B)
3) 2cosA cosB = cos (A+ B) + cos (A – B)
4) -2sinA sin B = cos (A + B) – cos (A – B)
or
2sinA sinB = cos (A – B) – cos (A + B)

Product formulae:

To express a sum or difference as a product, the above four formulae are not in a convenient form. To change them into convenient form, put A + B = C and A – B = D

Then, C + D = A + B + A – B

C + D = 2A

A = $\frac{C+D}{2}$

C - D = A + B - A + B

C - D = 2B B = $\frac{C-D}{2}$

∴ The product formulae are as follows:

1) sinC + sinD = 2sin$\frac{C+D}{2}$cos$\frac{C-D}{2}$
2) sinC – sinD = 2cos $\frac{C+D}{2}$sin$\frac{C-D}{2}$
3) cosC + cosD = 2cos$\frac{C+D}{2}$cos$\frac{C-D}{2}$
4) cosC – cosD = -2sin$\frac{C+D}{2}$sin$\frac{C-D}{2}$

WORKED EXAMPLES

PART – A

1)Express the following as sum or difference:

i)2cos70°sin10° ii)cos50°cos20° iii)2sin5xsinx

solution:

i)we know that, 2cosA sinB = sin (A + B) – sin (A – B)

2cos70°sin10° = sin(70° + 10°) - sin(70° – 10°)

$= \sin 80^\circ - \sin 60^\circ$

$= \sin 80^\circ - \frac{\sqrt{3}}{2}$

ii)we know that, 2cosA cosB = cos (A+ B) + cos (A – B)

$\cos 50^\circ \cos 20^\circ = \frac{1}{2}\ [cos(50^\circ + 20^\circ) + \cos(50^\circ - 20^\circ)]$

$= \frac{1}{2}\ [cos 70^\circ + \cos 30^\circ]$

$= \frac{1}{2}\left[cos 70^\circ + \frac{\sqrt{3}}{2}\right]$

iii)we know that, 2sinA sinB = cos (A – B) – cos (A + B)

2sin5xsinx = cos (5x-x) – cos(5x +x)

= cos4x – cos 6x

2)Express the following as products:

i)sin4x + sin2x ii) $\cos 70^\circ + \cos 50^\circ$

solution:

i)we know that, sinC + sinD = $2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

$\sin 4x + \sin 2x = 2\sin\frac{4x+2x}{2}\cos\frac{4x-2x}{2}$

$= 2\sin\frac{6x}{2}\cos\frac{2x}{2}$

= 2 sin3x cosx

ii)we know that, cosC + cosD = $2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

$\cos 70^\circ + \cos 50^\circ = 2\cos\frac{70^\circ+50^\circ}{2}\cos\frac{70^\circ-50^\circ}{2}$

$= 2\cos\frac{120^\circ}{2}\cos\frac{20^\circ}{2}$

$= 2\cos 60^\circ \cos 10^\circ$

$= 2\times\frac{1}{2}\cos 10^\circ$

$= \cos 10^\circ$

3) Prove that, $2\sin(45^\circ + A)\cos(45^\circ + B) = \cos(A + B) + \sin(A - B)$

Solution:

We know that, 2sinA cos B = sin(A + B) + sin (A – B)

$2\sin(45^\circ + A)\cos(45^\circ + B) = \sin(45^\circ + A + 45^\circ + B) + \sin(45^\circ + A - 45^\circ - B)$

$= \sin(90^\circ + (A + B)) + \sin(A - B)$

$= \cos(A + B) + \sin(A - B)$

PART – B

1)Prove that, $\cos 35^\circ + \cos 85^\circ + \cos 155^\circ = 0$

Solution:

We know that, $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

$\cos 35^\circ + \cos 85^\circ + \cos 155^\circ = 2\cos\frac{35^\circ + 85^\circ}{2}\cos\frac{35^\circ - 85^\circ}{2} + \cos 155^\circ$

$= 2\cos\frac{120^\circ}{2}\cos(\frac{-50^\circ}{2}) + \cos 155^\circ$

$= 2\cos 60^\circ \cos(-25^\circ) + \cos 155^\circ$

$= 2 \times \frac{1}{2}\cos 25^\circ + \cos 155^\circ$

$= \cos 25^\circ + \cos 155^\circ$

$= \cos 25^\circ + \cos(180^\circ - 25^\circ)$

$= \cos 25^\circ - \cos 25^\circ$

$= 0$

5)Prove that, $\sin 78^\circ - \sin 18^\circ + \cos 132^\circ = 0$

Solution:

We know that, $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

$\sin 78^\circ - \sin 18^\circ + \cos 132^\circ = 2\cos\frac{78^\circ + 18^\circ}{2}\sin\frac{78^\circ - 18^\circ}{2} + \cos 132^\circ$

$= 2\cos\frac{96^\circ}{2}\cos\frac{60^\circ}{2} + \cos 132^\circ$

$= 2\cos 48^\circ \sin(30^\circ) + \cos 132^\circ$

$= 2 \times \frac{1}{2}\cos 48^\circ + \cos 132^\circ$

$= \cos 48^\circ + \cos 132^\circ$

$= \cos 48^\circ + \cos(180^\circ - 48^\circ)$

$= \cos 48^\circ - \cos 48^\circ$

$= 0$

PART –C

1)Prove that, $(cosA - \cos B)^2 + (sinA - sinB)^2 = 4\, sin^2 \frac{A-B}{2}$

Solution:

We know that, sinC – sinD = $2\cos \frac{C+D}{2} \sin \frac{C-D}{2}$ and cosC – cosD = $-2\sin \frac{C+D}{2} \sin \frac{C-D}{2}$

$(cosA - \cos B)^2 + (sinA - sinB)^2 = \left(-2sin\frac{A+B}{2} sin\frac{A-B}{2}\right)^2 + \left(2cos\frac{A+B}{2} sin\frac{A-B}{2}\right)^2$

$= 4sin^2\frac{A+B}{2} sin^2\frac{A-B}{2} + 4cos^2\frac{A+B}{2}\ sin^2\frac{A-B}{2}$

$= 4sin^2\frac{A-B}{2}(\ sin^2\frac{A+B}{2} + cos^2\frac{A+B}{2})$

[since $sin^2\theta + cos^2\theta = 1$]

$= 4sin^2\frac{A-B}{2}$

2)If $\sin\alpha + \sin\beta$= a and $\cos\alpha + \cos\beta$ = b ,find $tan^2\frac{\alpha-\beta}{2}$

Solution:

Given: $\sin\alpha + \sin\beta$= a and $\cos\alpha + \cos\beta$ = b

We know that, sinC + sinD = $2\sin \frac{C+D}{2} \cos \frac{C-D}{2}$ and cosC + cosD = $2\cos\frac{C+D}{2} \cos\frac{C-D}{2}$

$2\sin \frac{\alpha+\beta}{2} \cos\frac{\alpha-\beta}{2}$ = a …….(1)

$2\cos\frac{\alpha+\beta}{2} \cos\frac{\alpha-\beta}{2}$ = b …….(2)

Squaring and adding (1) and (2)

$(2\sin \frac{\alpha+\beta}{2} \cos\frac{\alpha-\beta}{2})^2 + (2\cos \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2})^2 = a^2 + b^2$

$4sin^2\frac{\alpha+\beta}{2} cos^2\frac{\alpha-\beta}{2} + 4cos^2\frac{\alpha+\beta}{2}\ cos^2\frac{\alpha-\beta}{2} = a^2 + b^2$

$4cos^2\frac{\alpha-\beta}{2}(sin^2\frac{\alpha+\beta}{2} + cos^2\frac{\alpha+\beta}{2}) = a^2 + b^2$

$4cos^2\frac{\alpha-\beta}{2} = a^2 + b^2$ [since $sin^2\theta + cos^2\theta = 1$]

$cos^2 \frac{\alpha-\beta}{2} = \frac{a^2+b^2}{4}$

$sec^2 \frac{\alpha-\beta}{2} = \frac{4}{a^2+b^2}$ [since $\frac{1}{cos^2\theta} = sec^2\theta$]

$1 + tan^2 \frac{\alpha-\beta}{2} = \frac{4}{a^2+b^2}$ [since $sec^2\theta = 1 + tan^2\theta$]

$tan^2 \frac{\alpha-\beta}{2} = \frac{4}{a^2+b^2} - 1$

$= \frac{4-(a^2+b^2)}{a^2+b^2}$

$\therefore tan^2 \frac{\alpha-\beta}{2} = \frac{4-a^2-b^2}{a^2+b^2}$

3) Prove that, $(cosA + \cos B)^2 + (sinA - sinB)^2 = 4\, cos^2 \frac{A-B}{2}$

Solution:

We know that sinC – sinD = 2cos $\frac{C+D}{2}$sin$\frac{C-D}{2}$ and cosC + cosD = 2cos $\frac{C+D}{2}$cos$\frac{C-D}{2}$

$(cosA + \cos B)^2 + (sinA - sinB)^2 = \left(2cos\frac{A+B}{2}cos\frac{A-B}{2}\right)^2 + \left(2cos\frac{A+B}{2}sin\frac{A-B}{2}\right)^2$

$= 4cos^2\frac{A+B}{2}cos^2\frac{A-B}{2} + 4cos^2\frac{A+B}{2}\, sin^2\frac{A-B}{2}$

$= 4cos^2\frac{A+B}{2}(\, cos^2\frac{A-B}{2} + sin^2\frac{A-B}{2})$

[since $sin^2\theta + cos^2\theta = 1$]

$= 4cos^2\frac{A+B}{2}$

4) Prove that $\frac{sinA+sin2A+sin3A}{cosA+cos2A+cos3A}$ = tan2A

Solution:

$\frac{sinA+sin2A+sin3A}{cosA+cos2A+cos3A} = \frac{sinA+sin3A+sin2A}{cosA+cos3A+cos2A}$

$= \frac{2sin\left(\frac{A+3A}{2}\right)cos\left(\frac{A-3A}{2}\right) + sin2A}{2cos\left(\frac{A+3A}{2}\right)cos\left(\frac{A-3A}{2}\right)+cos2A}$

$= \frac{2sin\left(\frac{4A}{2}\right)cos\left(\frac{-2A}{2}\right) + sin2A}{2cos\left(\frac{4A}{2}\right)cos\left(\frac{-2A}{2}\right)+cos2A}$

$= \frac{2sin2A\cos(-A) + sin2A}{2cos2Acos(\,-A)+cos2A}$

$= \frac{2sin2A\ \text{cosA} + sin2A}{2cos2AcosA+cos2A}$ [since cos (-A) = cosA]

$= \frac{sin2A\ (\ 2\text{cosA} + 1\ )}{cos2A(\ 2cosA+1)}$

$= \frac{sin2A}{cos2A}$

= tan2A

<u>Exercise</u>

<u>PART-A</u>

1)Express the following as sum or difference .

i) 2sin7 θcos3 θ ii) -2sin3 θsin7 θ iii cos 15°sin 75° iv) sin3xcosx

2)Prove that 2sin($45^\circ + A$)cos(45° + B) = cos(A + B) + sin (A - B)

3) Express the following as products:

i) sin8A + sin2A ii) sin 20° - sin 40° iii) cos6 θ + cos2 θ iv) cos(x+h) –cosx

PART-B

Prove the following :

i) cos40° + cos80° + cos 160°= 0 ii) cos52° + cos68° + cos 172° = 0

iii)cos70°+ cos50° - cos 10°= 0 iv) cos20° + cos100°+cos 140°= 0 v) cos10° + cos70°=$\sqrt{3}$cos 40°

PART-C

1) Prove that, $(cos\alpha + \cos\beta)^2$ + $(sin\alpha + sin\beta)^2$ = 4 $cos^2 \frac{\alpha-\beta}{2}$

2) If sinα + sinβ= a and cosα + cosβ = b, show that $sec^2 \frac{\alpha-\beta}{2} = \frac{4}{a^2+b^2}$

3) Prove that, $(cos\alpha - \cos\beta)^2$ + $(sin\alpha + sin\beta)^2$ = 4 $sin^2 \frac{\alpha+\beta}{2}$

UNIT III DIFFERENTIAL CALCULUS

3.1 Limits

Introduction

Mathematical quantities can be divided into (i) constants (ii) Variables

Constants:

The quantity whose value does not change is called a constant.

Constants can be divided into (i) Absolute constants (ii) Arbitrary constants

Absolute constants are those whose value remains the same any time and at any place.

Arbitrary constants are those whose value changes from problem to problem. They are represented as a, b, c, f, g, h, ... etc.

Examples:

(i) The values 5, -7, 3, -11,π, e,... etc., are absolute constant

(i) ax + by + c = 0 here a, b, and c are arbitrary constants

(ii) $x^2 + y^2 + 2gx + 2fy + c = 0$ here g, f and c are arbitrary constants.

Variables

The quantity, whose value changes is called a variable they are represented as u, v, w, x, y, z, θ,.. Variables can be divided into (i) Independent variable (ii) Dependent variable.

In the equation $y = x^2$, x is called independent variable and y is called dependent variable.

The relation between two variables is called a function and it is denoted by y = f(x).

Example: $y = f(x) = x^2$ or sin x.

<u>Limit</u>

The value of f(x) at x = a is f(a) and is called the functional value of f(x) at x = a.

x → a means 'x' approaches to 'a' (both from L.H.S to R.H.S) but not equal to 'a'

[x is very nearer to 'a']

Let $y = f(x) = \frac{x^2-4}{x-2}$

Now, let us study the behaviour of f(x) as x→2 here, the function value of f(x) at x = a does not exist $f(2) = \frac{2^2-4}{2-2} = \frac{0}{0}$ (indeterminat form)

<u>Left Hand limit:</u>

As x → 2 from left hand side, f(x)→ 4

$$\lim_{x\to 2^-} \frac{x^2-4}{x-2} = 4$$

<u>Right Hand limit:</u>

As x → 2 from Right hand side, f(x)→ 4

$$\lim_{x\to 2^+} \frac{x^2-4}{x-2} = 4$$

X	.....	1.9	1.99	1.999	2.001	2.01	2.1	.....
Y= $\frac{x^2-4}{x-2}$	.....	3.9	3.99	3.999	4.001	4.01	4.1	.....

From the above table, it is clear that as x approaches 2, f(x) approach to 4.

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$$

Let y = f(x) be any function, then f(x) approaches a constant 'l' as x approach to 'a' is called limit value if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a} f(x) = l$

Results:

1) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a}[f(x)] \pm \lim_{x\to a}[g(x)]$

2) $\lim_{x\to a}[kf(x)] = k \lim_{x\to a}[f(x)]$, where k is a constant

3) $\lim_{x\to a}[f(x).g(x)] = \lim_{x\to a}[f(x)].\lim_{x\to a}[g(x)]$

4) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a}[f(x)]}{\lim_{x\to a}[g(x)]}$, Provided $\lim_{x\to a} g(x) \neq 0$

Formulae:

1) $\lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}$, for all values of x.

2) $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

3) $\lim_{\theta\to 0} \frac{\tan\theta}{\theta} = 1$

Note:

1) $\lim_{\theta\to 0} \frac{\sin n\theta}{\theta} = \lim_{\theta\to 0} \frac{\sin n\theta}{n\theta} n$

$= n(\lim_{\theta\to 0} \frac{\sin n\theta}{n\theta})$

$= n(\lim_{n\theta\to 0} \frac{\sin n\theta}{n\theta})$

= n(1)

= n

2) $\lim_{\theta\to 0} \frac{\tan n\theta}{\theta} = n$

WORKED EXAMPLES
PART – A

1. Evaluate: $\lim_{x\to 0}\frac{3x^2+2x+1}{5x^2+6x+7}$

Solution:

$$\lim_{x\to 0}\frac{3x^2+2x+1}{5x^2+6x+7} = \frac{3(0)+2(0)+1}{5(0)+6(0)+7} = \frac{1}{7}$$

2. Evaluate: $\lim_{x\to 0}\frac{3x^2+4x}{5x-7x^2}$

Solution:

$$\text{Lim}_{x\to 0}\frac{3x^2+4x}{5x-7x^2} = \lim_{x\to 0}\frac{x(3x+4)}{x(5-7x)}$$

$$= \lim_{x\to 0}\frac{(3x+4)}{(5-7x)}$$

$$= \frac{(0+4)}{(5-0)} = \frac{4}{5}$$

3. Evaluate: $\lim_{\theta\to 0}\frac{\sin 7\theta}{\theta}$

Solution:

$$\lim_{\theta\to 0}\frac{\sin 7\theta}{\theta} = \lim_{\theta\to 0}\frac{\sin 7\,\theta}{7\theta}.7$$

$$= 7(\lim_{\theta\to 0}\frac{\sin 7\,\theta}{7\theta})$$
$$= 7(1)$$
$$= 7$$

4. Evaluate: $\lim_{x\to 1}\frac{lx^2+mx+n}{ax^2+bx+c}$

Solution:

$$\lim_{x\to 1}\frac{lx^2+mx+n}{ax^2+bx+c} = \frac{l(1)^2+m(1)+n}{a(1)^2+b(1)+c}$$

$$= \frac{l+m+n}{a+b+c}$$

PART – B

1. Evaluate: $\lim_{x\to 1}\frac{x^2+x-2}{x^2-4x+3}$

Solution:

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-4x+3} = \frac{1+1-2}{1-4+3} = \frac{0}{0} \text{ Inderminent}$$

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-4x+3} = \lim_{x\to 1}\frac{(x+2)(x-1)}{(x-3)(x-1)}$$

$$= \lim_{x\to 1}\frac{x+2}{x-3}$$

$$= \frac{-3}{2}$$

2. Evaluate: $\lim_{x\to 1}\frac{x^2-3x+2}{x^2-5x+4}$

Solution:

$$\lim_{x\to 1}\frac{x^2-3x+2}{x^2-5x+4} = \lim_{x\to 1}\frac{(x-1)(x-2)}{(x-1)(x-4)}$$

$$= \lim_{x\to 1}\frac{(x-2)}{(x-4)}$$

$$= \frac{1-2}{1-4}$$

$$= \frac{1}{3}$$

3. Evaluate: $\lim_{x\to 2}\frac{x^{10}-2^{10}}{x-2}$

Solution:

$$\lim_{x\to 2}\frac{x^{10}-2^{10}}{x-2} = 10(2)^{10-1} \qquad [\text{since } \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}]$$

$$= 10(2)^9$$

$$= 10(512) = 5120$$

4. Evaluate: $\lim_{x\to 3}\frac{x^5-3^5}{x-3}$

Solution:

$$\lim_{x\to 3}\frac{x^5-3^5}{x-3} = 5(3)^{5-1} \qquad [\text{since } \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}]$$

$$= 5(3)^4$$

$$= 5(81) = 405$$

5. Evaluate: $\lim_{x\to 2}\frac{x^5-2^5}{x-2}$

Solution:

$$\lim_{x\to 2}\frac{x^5-2^5}{x-2} = 5(2)^{5-1} \qquad [\text{since } \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}]$$

$$= 5(2)^4$$

$$= 5(16) = 80$$

6. Evaluate: $\lim_{x\to 3}\frac{x^4-81}{x-3}$

Solution:

$$\lim_{x\to 3}\frac{x^4-81}{x-3} = \lim_{x\to 3}\frac{x^4-3^4}{x-3}$$

$= 4(3)^{4-1}$ [since $\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$]

$= 4(3)^3 = 4(27) = 108$

7. Evaluate: $\lim_{x\to 4}\frac{x^3-64}{x-4}$

Solution:

$$\lim_{x\to 4}\frac{x^3-64}{x-4} = \lim_{x\to 4}\frac{x^3-4^3}{x-4}$$

$= 3(4)^{3-1}$ [since $\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$]

$= 3(4)^2$

$= 3(16) = 48$

8. Evaluate: $\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a}$

Solution*:*

$$\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a} = \lim_{x\to a}\frac{x^{\frac{1}{2}}-a^{\frac{1}{2}}}{x-a}$$

$$= \frac{1}{2}.(a)^{\frac{1}{2}-1}$$

$$= \frac{1}{2}.a^{\frac{-1}{2}} = \frac{1}{2}.\frac{1}{a^{\frac{1}{2}}} = \frac{1}{2\sqrt{a}}$$

9. Evaluate: $\lim_{x\to 0}\frac{\sin 5x}{4x}$

Solution:

$$\lim_{x\to 0}\frac{\sin 5x}{4x} = \frac{1}{4}\lim_{x\to 0}\frac{\sin 5x}{x}$$

$$= \frac{5}{4}\left(\lim_{x\to 0}\frac{\sin 5x}{5x}\right)$$

$$= \frac{5}{4}(1)$$

$$= \frac{5}{4}$$

10. Evaluate: $\lim_{x\to 0}\frac{tan^2 5x}{x^2}$

Solution:

$$\lim_{x\to 0}\frac{tan^2 5x}{x^2} = \lim_{x\to 0}\left(\frac{\tan 5x}{x}\right)^2$$

$$= \lim_{x\to 0} \frac{\tan 5x}{x} . \frac{\tan 5x}{x}$$

$$= \lim_{x\to 0} \frac{\tan 5x}{x} . \lim_{x\to 0} \frac{\tan 5x}{x}$$

$$= 5(5) = 25$$

PART – C

1. Evaluate: $\lim_{x\to 3} \frac{x^5-243}{x^2-9}$

Solution:

$$\lim_{x\to 3} \frac{x^5-243}{x^2-9} = \lim_{x\to 3} \frac{x^5-3^5}{x^2-3^2}$$

$$= \lim_{x\to 3} \frac{\frac{x^5-3^5}{x-3}}{\frac{x^2-3^2}{x-3}}$$

$$= \frac{\lim_{x\to 3} \frac{x^5-3^5}{x-3}}{\lim_{x\to 3} \frac{x^2-3^2}{x-3}}$$

$$= \frac{5(3)^{5-1}}{2(3)^{2-1}}$$

$$= \frac{5(3)^4}{2(3)^1}$$

$$= \frac{135}{2}$$

2. Evaluate: $\lim_{x\to 3} \frac{x^4-81}{x^3-27}$

Solution:

$$\lim_{x\to 3} \frac{x^4-81}{x^3-27} = \lim_{x\to 3} \frac{x^4-3^4}{x^3-3^3}$$

$$= \lim_{x\to 3} \frac{\frac{x^4-3^4}{x-3}}{\frac{x^3-3^3}{x-3}}$$

$$= \frac{\lim_{x\to 3} \frac{x^4-3^4}{x-3}}{\lim_{x\to 3} \frac{x^3-3^3}{x-3}}$$

$$= \frac{4(3)^{4-1}}{3(3)^{3-1}}$$

$$= \frac{4(3)^3}{3(3)^2}$$

$$= 4$$

3. Evaluate : $\lim_{x\to 2}\frac{x^5-32}{x^3-8}$

Solution:

$$\lim_{x\to 2}\frac{x^5-32}{x^3-8} = \lim_{x\to 2}\frac{x^5-2^5}{x^3-2^3}$$

$$= \lim_{x\to 2}\frac{\frac{x^5-2^5}{x-2}}{\frac{x^3-2^3}{x-2}}$$

$$= \frac{\lim_{x\to 2}\frac{x^5-2^5}{x-2}}{\lim_{x\to 2}\frac{x^3-2^3}{x-2}}$$

$$= \frac{5(2)^{5-1}}{3(2)^{3-1}}$$

$$= \frac{5(2)^4}{3(2)^2}$$

$$= \frac{20}{3}$$

4. Evaluate: $\lim_{\theta\to 0}\frac{7\sin 5\theta}{6\sin 8\theta}$

Solution:

$$\lim_{\theta\to 0}\frac{7\sin 5\theta}{6\sin 8\theta} = \frac{7}{6}\lim_{\theta\to 0}\frac{\sin 5\theta}{\sin 8\theta}$$

$$= \frac{7}{6}\lim_{\theta\to 0}\frac{\frac{\sin 5\theta}{\theta}}{\frac{\sin 8\theta}{\theta}}$$

$$= \frac{7}{6}\frac{\lim_{\theta\to 0}\frac{\sin 5\theta}{\theta}}{\lim_{\theta\to 0}\frac{\sin 8\theta}{\theta}}$$

$$= \frac{7}{6}\left(\frac{5}{8}\right)$$

$$= \frac{35}{48}$$

5. Evaluate: $\lim_{\theta\to 0}\frac{5\sin 6\theta}{3\sin 2\theta}$

Solution:

$$\lim_{\theta\to 0}\frac{5\sin 6\theta}{3\sin 2\theta} = \frac{5}{3}\lim_{\theta\to 0}\frac{\sin 6\theta}{\sin 2\theta}$$

$$= \frac{5}{3}\lim_{\theta\to 0}\frac{\frac{\sin 6\theta}{\theta}}{\frac{\sin 2\theta}{\theta}}$$

$$= \frac{5}{3}\frac{\lim_{\theta\to 0}\frac{\sin 6\theta}{\theta}}{\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta}}$$

$$= \frac{5}{3}\left(\frac{6}{2}\right)$$

$$= 5$$

6. Evaluate: $\lim_{\theta \to 0} \frac{\tan 6\theta}{13\theta}$

Solution:

$$\lim_{\theta \to 0} \frac{\tan 6\theta}{13\theta} = \frac{1}{13} \lim_{\theta \to 0} \frac{\tan 6\theta}{\theta}$$

$$= \frac{1}{13}(6)$$

$$= \frac{6}{13}$$

EXERCISE

PART-A

1. Evaluate: $\lim_{x \to 4} \frac{x^2-7x+12}{x^2-3x-4}$
2. Evaluate: $\lim_{x \to 1} \frac{4x+5}{3x-7}$

PART-B

1. Evaluate: $\lim_{x \to a} \frac{x^{2/7}-a^{2/7}}{x-a}$
2. Evaluate: $\lim_{x \to 0} \frac{\sin 8x}{2x}$
3. Evaluate: $\lim_{x \to 0} \frac{\sin 6x}{\sin 7x}$
4. Evaluate: $\lim_{x \to 2} \frac{x^3-8}{x-2}$
5. Evaluate: $\lim_{x \to 2} \frac{x^5-32}{x-2}$

PART-C

1. Evaluate: $\lim_{x \to a} \frac{x^{5/8}-a^{5/8}}{x^{1/3}-a^{1/3}}$
2. Evaluate: $\lim_{x \to 3} \frac{x^4-81}{x^2-9}$

3.2 DIFFERENTIATION

Let y = f(x) be a function of x. Let Δx be a small increment in x and let Δy be the corresponding increment in y.

∴ y + Δy = f(x + Δx)

Δy = f(x + Δx) - y

Δy = f(x + Δx) – f(x)

$$\therefore \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Taking the limit as Δx → 0,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ is called the differential coefficient of y with respect to x and is denoted as $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

List of formulae:

(1) $\frac{d}{dx}(x^n) = nx^{n-1}$, where n is a real number

(2) $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$

(3) $\frac{d}{dx}(e^x) = e^x$

(4) $\frac{d}{dx}(\log x) = \frac{1}{x}$

(5) $\frac{d}{dx}(\sin x) = \cos x$

(6) $\frac{d}{dx}(\cos x) = -\sin x$

(7) $\frac{d}{dx}(\tan x) = \sec^2 x$

(8) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(9) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(10) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

(11) $\frac{d}{dx}(k) = 0$, where k is some constant

Results:

1) If u and v are functions of x,

   $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

2) $\frac{d}{dx}(ku) = k\frac{du}{dx}$, where k is a constant.

3) **Product Rule of Differentiation**: If u and v are functions of x, then

   $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

4) $\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$, where u, v and w are functions of x

5) **Quotient Rule of Differentiation:** If u and v are functions of x, then

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

WORKED EXAMPLES

PART – A

1. Find $\frac{dy}{dx}$, if (i) $y = \frac{1}{x^2}$ (ii) $y = x^3 + 2$ (iii) $y = \frac{1}{\sqrt{x}}$

(iv) $y = \sqrt{x}$ (v) $y = \frac{1}{\sin x}$ (vi) $y = 9\sqrt{x} + x^2$

(vii) $y = e^x + \tan x$ (viii) $y = \frac{1}{\cos x}$ (ix) $y = \cot x + \log x$

Solution:

(i) $y = \frac{1}{x^2} = x^{-2}$

$$\frac{dy}{dx} = \frac{d}{dx}(x^{-2})$$

$$= -2x^{-3} = -\frac{2}{x^3}$$

(ii) $y = x^3 + 2$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 + 2)$$

$$= 3x^2 + 0 = 3x^2$$

(iii) $y = \frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}} = x^{-1/2}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{-1/2}\right) = \frac{-1}{2}x^{-\frac{1}{2}-1}$$

$$= \frac{-1}{2}x^{-3/2}$$

(iv) $y = \sqrt{x} = x^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{\frac{1}{2}-1}$$

$$= \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$$

(v) $y = \frac{1}{\sin x} = \text{cosec } x$

$$\frac{dy}{dx} = \frac{d}{dx}(\text{cosec } x) = -cosecx \cot x$$

(vi) $y = 9\sqrt{x} + x^2$

$$\frac{dy}{dx} = \frac{d}{dx}\left(9\sqrt{x} + x^2\right) = 9\frac{d}{dx}\left(\sqrt{x}\right) + \frac{d}{dx}(x^2)$$

$$= 9\frac{1}{2\sqrt{x}} + 2x$$

(vii) $y = e^x + \tan x$

$$\frac{dy}{dx} = \frac{d}{dx}(e^x + \tan x) = e^x + sec^2x$$

(vii) $y = \frac{1}{\cos x} = \sec x$

$$\frac{dy}{dx} = \frac{d}{dx}(\sec x) = \sec x \tan x$$

(ix) y = cot x + log x

$$\frac{dy}{dx} = \frac{d}{dx}(\cot x + \log x) = -\text{cosec}^2 x + \frac{1}{x}$$

PART – B

1. Find $\frac{dy}{dx}$, if $y = \frac{3}{x^2} + \frac{2}{X} + \frac{1}{4}$

Solution:

$$y = \frac{3}{x^2} + \frac{2}{X} + \frac{1}{4} = 3x^{-2} + 2x^{-1} + \frac{1}{4}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(3x^{-2} + 2x^{-1} + \frac{1}{4}\right) = 3(-2)x^{-3} + 2(-1)x^{-2}$$

$$= \frac{-6}{x^3} - \frac{2}{x^2}$$

2. Find $\frac{dy}{dx}$, if $y = \frac{4}{x^2} - \frac{3}{X} + \frac{1}{2}$

Solution:

$$y = \frac{4}{x^2} - \frac{3}{X} + \frac{1}{2} = 4x^{-2} - 3x^{-1} + \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{d}{dx}(4x^{-2} - 3x^{-1} + \frac{1}{2}) = 4(-2)x^{-2-1} - 3(-1)x^{-1-1}$$

$$= -8x^{-3} + 3x^{-2}$$

3. If y = 3log x - $4\sqrt{x} + e^x$, find $\frac{dy}{dx}$

Solution:

y = 3log x - $4\sqrt{x} + e^x$

$$\frac{dy}{dx} = \frac{d}{dx}(3\log x - 4\sqrt{x} + e^x) = 3\frac{1}{x} - 4\frac{1}{2\sqrt{x}} + e^x$$

4. If y = $\frac{4}{x} + 7\cot x - 9\log x - \frac{8}{\cos x} + 11$, find $\frac{dy}{dx}$

Solution:

$$y = \frac{4}{x} + 7\cot x - 9\log x - \frac{8}{\cos x} + 11 = 4x^{-1} + 7\cot x - 9\log x - 8\sec x + 11$$

$$\frac{dy}{dx} = \frac{d}{dx}(4x^{-1} + 7\cot x - 9\log x - 8\sec x + 11)$$

$$= 4(-1)x^{-1-1} + 7(-\text{cosec}^2 x) - 9\frac{1}{x} - 8\sec x \tan x$$

$$= -4x^{-2} - 7\,\text{cosec}^2 x - \frac{9}{x} - 8\sec x \tan x$$

5. If y = $\sqrt{x} - 3e^x + 7\sin x - \frac{8}{\sec x} + 11$, find $\frac{dy}{dx}$

Solution:

y =$\sqrt{x} - 3e^x + 7\sin x - \frac{8}{\sec x} + 11 = \sqrt{x} - 3e^x + 7\sin x - 8\cos x + 11$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x} - 3e^x + 7\sin x - 8\cos x + 11\right)$$

$$= \frac{1}{2\sqrt{x}} - 3e^x + 7\cos x + 8\sin x$$

6. Find $\frac{dy}{dx}$, if $y = e^x \sin x$

Solution:

$y = e^x \sin x$

$\frac{dy}{dx} = \frac{d}{dx}(e^x \sin x)$

We know that, $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = e^x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(e^x)$$

$$\frac{dy}{dx} = e^x \cos x + e^x \sin x$$

7. Find $\frac{dy}{dx}$, if $y = \tan x \log x$

Solution:

Y= tan x log x

$\frac{dy}{dx} = \frac{d}{dx}(\tan x \log x)$

We know that $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = \tan x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\tan x)$$

$$\frac{dy}{dx} = \tan x \frac{1}{x} + \log x \sec^2 x$$

8. Find, $\frac{dy}{dx}$,if $y = x^2 \sec x$

Solution:

$y = x^2 \sec x$

$\frac{dy}{dx} = \frac{d}{dx}(x^2 \sec x)$

We know that $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sec x) + \sec x \frac{d}{dx}x^2$$

$$\frac{dy}{dx} = x^2 \sec x \tan x + 2x \sec x$$

9. Find, $\frac{dy}{dx}$ if $y = x^2e^x \sin x$

Solution:

$y = x^2e^x \sin x$

$\frac{dy}{dx} = \frac{d}{dx}(x^2e^x \sin x)$

We know that $\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$

$\frac{dy}{dx} = \frac{d}{dx}(x^2e^x \sin x)$

$= x^2e^x \frac{d}{dx}(\sin x) + x^2 \sin x \frac{d}{dx}(e^x) + e^x \sin x \frac{d}{dx}(x^2)$

$\frac{dy}{dx} = x^2e^x \cos x + x^2e^x \sin x + 2x.e^x \sin x$

10. Find $\frac{dy}{dx}$, if $y = x^2 \cot x \log x$

Solution:

$y = x^2 \cot x \ \log x$

$\frac{dy}{dx} = \frac{d}{dx}(x^2 \cot x \ \log x)$

We know that $\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$

$\frac{dy}{dx} = \frac{d}{dx}(x^2 \cot x \log x)$

$= x^2 \cot x \frac{d}{dx}(\log x) + x^2 \log x \frac{d}{dx}(\cot x) + \cot x \log x \frac{d}{dx}(x^2)$

$\frac{dy}{dx} = x^2 \cot x \ \frac{1}{x} - x^2 \log x \operatorname{cosec}^2 x + 2x.\cot x \log x$

11. Find $\frac{dy}{dx}$, if $y = \frac{\cos x}{\sqrt{x}}$

Solution:

$y = \frac{\cos x}{\sqrt{x}}$

We know that $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{dy}{dx} = \frac{\sqrt{x}\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2}$

$$\frac{dy}{dx} = \frac{(\sqrt{x})[-\sin x] - \cos x \frac{1}{2\sqrt{x}}}{x}$$

PART – C

1. Find $\frac{dy}{dx}$, if $y = (x^2 + 3)\cos x \log x$

Solution:

$y = (x^2 + 3)\cos x \log x$

$\frac{dy}{dx} = \frac{d}{dx}[(x^2 + 3)\cos x \log x]$

We know that $\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$

$\frac{dy}{dx} = \frac{d}{dx}[(x^2 + 3)\cos x \log x]$

$= (x^2 + 3)\cos x\frac{d}{dx}(\log x) + (x^2 + 3)\log x\frac{d}{dx}(\cos x) + \cos x \log x\frac{d}{dx}(x^2 + 3)$

$\frac{dy}{dx} = (x^2 + 3)\cos x\frac{1}{x} + (x^2 + 3)\log x\,(-\sin x) + \cos x \log x\,.\,2x$

2. Find $\frac{dy}{dx}$, if $y = \frac{1+x+x^2}{1-x+x^2}$

Solution:

$y = \frac{1+x+x^2}{1-x+x^2}$

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1+x+x^2}{1-x+x^2}\right)$

We know that $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\frac{dy}{dx} = \frac{(1-x+x^2)\frac{d}{dx}(1+x+x^2)-(1+x+x^2)\frac{d}{dx}(1-x+x^2)}{(1-x+x^2)^2}$

$= \frac{(1-x+x^2)(1+2x)-(1+x+x^2)(-1+2x)}{(1-x+x^2)^2}$

$= \frac{2-2x^2}{(1-x+x^2)^2}$

$= \frac{2(1-x^2)}{(1-x+x^2)^2}$

3. Find $\frac{dy}{dx}$, if $y = \frac{e^x \sin x}{x^2+1}$

Solution:

$y = \frac{e^x \sin x}{x^2+1}$

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x \sin x}{x^2+1}\right)$

We know that $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{(x^2+1)\frac{d}{dx}(e^x \sin x)-(e^x \sin x)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)\left[e^x\frac{d}{dx}(\sin x)+(\sin x)\frac{d}{dx}(e^x)\right]-e^x \sin x.2x}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{(x^2+1)[e^x \cos x+(\sin x)(e^x)] - 2x\, e^x \sin x}{(x^2+1)^2}$$

4. Find $\frac{dy}{dx}$, if $y = \frac{x^3 \tan x}{e^x+1}$

Solution:

$$y = \frac{x^3 \tan x}{e^x+1}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^3 \tan x}{e^x+1}\right)$$

We know that $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{(e^x+1)\frac{d}{dx}(x^3 \tan x)-(x^3 \tan x)\frac{d}{dx}(e^x+1)}{(e^x+1)^2}$$

$$= \frac{(e^x+1)\left[x^3\frac{d}{dx}(\tan x)+(\tan x)\frac{d}{dx}(x^3)\right]-x^3 \tan x.e^x}{(e^x+1)^2}$$

$$\frac{dy}{dx} = \frac{(e^x+1)\left[x^3\sec^2 x + \tan x(3x^2)\right]-x^3 \tan x.e^x}{(e^x+1)^2}$$

EXERCISE

PART-A

1. Find $\frac{dy}{dx}$, if $y = \frac{3}{x^2} + \frac{2}{X} + \frac{1}{4}$
2. Find $\frac{dy}{dx}$, if $y = \frac{1}{x^3} + \frac{7}{X} + \operatorname{cosec} x$
3. Find $\frac{dy}{dx}$, if $y = e^x + \frac{1}{\cos x} + \log x$
4. Find $\frac{dy}{dx}$,if $y = \sin 3x + \tan 3x$
5. Find $\frac{dy}{dx}$, if $y = \frac{3}{x} + \frac{1}{4}$

PART-B

6. Find $\frac{dy}{dx}$, if $y = e^{3x} \cos x$
7. Find $\frac{dy}{dx}$, if $y = x^2 e^{5x} \log x$
8. Find $\frac{dy}{dx}$, if $y = (x^2 + 3) \cos x \log x$
9. Find $\frac{dy}{dx}$, if $y = \frac{\cos x}{\sqrt{x}}$

PART-C

10. Find $\frac{dy}{dx}$, if $y = \frac{1+2x-3x^2}{1+x+x^2}$

11. Find $\frac{dy}{dx}$, if $y = \frac{x^3 \cot x}{e^x+1}$

3.3 DIFFERENTIATION METHODS

Function of Funtion Rule:

If y = f(u) and u = f(x), then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

It is called function of function rule. This rule can be extended which is known as 'Chain rule'.

Chain rule:

If y = f(u), u = f(v) and v = f(x), then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx}$

WORKED EXAMPLES

PART –A

Find $\frac{dy}{dx}$ if

1) $y = (2x + 5)^3$ 2) $y = \sqrt{sin\ x}$ 3) $y = \cos^4 x$ 4) $y = e^{\tan x}$

5) y = log (sec x) 6) y = sin mx 7) $y = \sec \sqrt{x}$ 8) y = cos (2 – 3x)

9) $y = \sin^2 x$ 10) $y = \tan^4 x$ 11) $y = e^{x^2}$ 12) y = log(sec x + tan x)

Solution:

1. $y = (2x + 5)^3$

$y = u^3$, where u = 2x + 5

$\frac{dy}{du} = 3u^2 and\ \frac{du}{dx} = 2$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

$$\frac{dy}{dx} = 3(u)^2(2)$$

$$\frac{dy}{dx} = 6u^2 = 6(2x + 3)^2$$

2. $y = \sqrt{sin\ x}$

$y = \sqrt{u}$, where u = sin x

$\frac{dy}{du} = \frac{1}{2\sqrt{u}}\ and\ \frac{du}{dx} = \cos x$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

$$= \frac{1}{2\sqrt{u}}.\cos x$$

$$\frac{dy}{dx} = \frac{cos\ x}{2\sqrt{\sin x}}$$

3. $y = \cos^4 x$

Solution:

$y = u^4, where\ u = \cos x$

$\frac{dy}{du} = 4u^3$ and $\frac{du}{dx} = -\sin x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= 4u^3(-\sin x)$$

$$\frac{dy}{dx} = -4\cos^3 x \sin x$$

4. $y = e^{\tan x}$

Solution:

$y = e^u$, where $u = \tan x$

$$\frac{dy}{du} = e^u \ and \ \frac{du}{dx} = sec^2 x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= e^u(\sec^2 x)$$

$$\frac{dy}{dx} = e^{\tan x}(\sec^2 x)$$

5. $y = \log(\sec x)$

Solution:

$y = \log u$, where $u = \sec x$

$$\frac{dy}{du} = \frac{1}{u} \ and \ \frac{du}{dx} = \sec x \tan x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{u} \sec x tan\ x$$

$$= \frac{1}{\sec x}(\sec x \tan x)$$

$$\frac{dy}{dx} = \tan x$$

6. $y = \sin mx$

Solution:

$y = \sin u$, where $u = mx$

$$\frac{dy}{du} = \cos u \ and \ \frac{du}{dx} = m$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \cos u \ (m)$$

$$\frac{dy}{dx} = m \cos mx$$

7. $y = \sec(\sqrt{x})$

Solution:

$y = \sec(u)$, where $u = \sqrt{x}$

$\frac{dy}{du} = \sec u \tan u$ and $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$

$= (\sec u \tan u) \left(\frac{1}{2\sqrt{x}}\right)$

$\frac{dy}{dx} = (\frac{1}{2\sqrt{x}} . \sec\sqrt{x} . \tan\sqrt{x})$

8. y = cos (2 – 3x)

Solution:

y = cos u, where u = 2 - 3x

$\frac{dy}{du} = -\sin u \ and \ \frac{du}{dx} = -3$

$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$

$\frac{dy}{dx} = -\sin u(-3)$

= 3sin u

$\frac{dy}{dx}$ = 3sin(2-3x)

9. y = $\sin^2 x$

Solution:

y = u^2 , where u = sin x

$\frac{dy}{du} = 2u \ and \ \frac{du}{dx} = cosx$

$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$

$= 2u \cos x$

$\frac{dy}{dx}$ = 2sin x cos x

10. y = $\tan^4 x$

Solution:

y = u^4 , where u = tan x

$\frac{dy}{du} = 4u^3 \ and \ \frac{du}{dx} = sec^2 x$

$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$

$= 4u^3 \sec^2 x$

$\frac{dy}{dx} = 4tan^3 x sec^2 x$

11. y = e^{x^2}

Solution:

y = e^u , where u = x^2

$\frac{dy}{du} = e^u \; and \; \frac{du}{dx} = 2x$

$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$

$= e^u 2x$

$\frac{dy}{dx} = 2xe^{x^2}$

12. y = log(sec x + tan x)

Solution:

y = log(u), where u = sec x + tan x

$\frac{dy}{du} = \frac{1}{u} \; and \; \frac{du}{dx} = \sec x tan\, x + sec^2 x$

$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$

$\frac{dy}{dx} = \frac{1}{u}(\sec x \tan x + sec^2 x)$

$= \frac{1}{\sec x + \tan x}(\sec x \tan x + sec^2 x)$

$\frac{dy}{dx} = \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} = \sec x$

Note:

1. $\frac{d}{dx}(e^{mx}) = me^{mx}$ 2. $\frac{d}{dx}$ (sin mx) = m cos mx 3. $\frac{d}{dx}$ (cos mx) = – m sin mx

4. $\frac{d}{dx}$ (tan mx) = m $\sec^2$mx 5. $\frac{d}{dx}$ (cot mx) = -m $cosec^2$mx 6. $\frac{d}{dx}$ (sec mx) = m sec mx tan mx

7. $\frac{d}{dx}$ (cosec mx) = - m cosecmx cot mx

PART – B

1. If y = sin(e^x log x), then find $\frac{dy}{dx}$

Solution:

y = sin(e^x log x)

$\frac{dy}{dx}$ = cos(e^x log x)$\frac{d}{dx}(e^x \log x)$ $\quad since \; \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx}$ = cos(e^x log x)($e^x \frac{1}{x} + \log x e^x$)

2. If y = e^{3x} sin 2x log x, then find $\frac{dy}{dx}$

Solution:

$y = e^{3x} \sin 2x \log x$

$u = e^{3x}$; $v = \sin 2x$; $w = \log x$

$\frac{du}{dx} = 3e^{3x}$; $\frac{dv}{dx} = 2\cos 2x$; $\frac{dv}{dw} = \frac{1}{x}$

We know that $\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$

$$\frac{dy}{dx} = e^{3x}\sin 2x\ \frac{1}{x} + \log x\ e^{3x}\ 2\cos 2x + \sin 2x \log x\ 3e^{3x}$$

3.if $y = \log\left(\frac{1-\cos x}{1+\cos x}\right)$, then find $\frac{dy}{dx}$

Solution:

$$\frac{dy}{dx} = \left(\frac{1}{\frac{1-\cos x}{1+\cos x}}\right)\frac{d}{dx}\left(\frac{1-\cos x}{1+\cos x}\right)$$

$$\frac{dy}{dx} = \left(\frac{1+\cos x}{1-\cos x}\right)\frac{(1+\cos x)(\sin x)-(1-\cos x)(-\sin x)}{(1+\cos x)^2}$$

$$\frac{dy}{dx} = \left(\frac{1+\cos x}{1-\cos x}\right)\frac{(\sin x+\cos x\sin x+\sin x-\cos x\sin x)}{(1+\cos x)^2}$$

$$\frac{dy}{dx} = \frac{2\sin x(1+\cos x)}{(1+\cos x)^2(1-\cos x)}$$

$$\frac{dy}{dx} = \frac{1}{(1-\cos x)}\frac{2\sin x}{(1+\cos x)}$$

$$\frac{dy}{dx} = \frac{2\sin x}{1-cos^2 x}$$

$$\frac{dy}{dx} = \frac{2\sin x}{sin^2 x} = \frac{2}{\sin x}$$

$$\frac{dy}{dx} = 2\ \text{cosec}\ x$$

Successive differentiation

If y = f(x) be any function then,

i) First derivative of y w.r.to x is denoted by $\frac{dy}{dx}$ (or) y_1 (or) y' (or) f'(x) (or) Dy where $D = \frac{d}{dx}$

ii) Second derivative of y w.r.to x is denoted by $\frac{d^2y}{dx^2}$ (or) y_2 (or) y''(or) f''(x) (or) D^2y where $D^2 = \frac{d^2}{dx^2}$

Worked Example

PART – B

1. If $y = e^{3x}$, then find $\frac{d^2y}{dx^2}$ or y_2 or y''

Solution:

$y = e^{3x}$

$\frac{dy}{dx} = 3e^{3x}$

$\frac{d^2y}{dx^2} = 3(3)\ e^{3x}$

$\frac{d^2y}{dx^2} = 9\ e^{3x}$

2. If y = sin 4x, then find $\frac{d^2y}{dx^2}$ or y_2 or y''

Solution:

$y = \sin 4x$

$\frac{dy}{dx} = 4\cos 4x$

$\frac{d^2y}{dx^2} = 4(4)(-\sin 4x)$

$\frac{d^2y}{dx^2} = -16 \sin 4x$

3. If y = cos 5x, then find $\frac{d^2y}{dx^2}$ or y_2 or y''

Solution:

$y = \cos 5x$

$\frac{dy}{dx} = -5\sin 5x$

$\frac{d^2y}{dx^2} = -5(5)(\cos 5x)$

$\frac{d^2y}{dx^2} = -25 \cos 5x$

4. If y = tan x, then find $\frac{d^2y}{dx^2}$ or y_2 or y''

Solution:

$y = \tan x$

$\frac{dy}{dx} = sec^2x$

$\frac{d^2y}{dx^2} = 2\sec x \sec x \tan x$

$\frac{d^2y}{dx^2} = 2 \sec^2 x \tan x$

5. If $y = Ae^{3x} + Be^{-3x}$, then find $\frac{d^2y}{dx^2}$ or y_2 or y''

Solution:

$y = Ae^{3x} + Be^{-3x}$

$\frac{dy}{dx} = 3Ae^{3x} - 3Be^{-3x}$

$\frac{d^2y}{dx^2} = 3(3)Ae^{3x} - 3(-3)Be^{-3x}$

$\frac{d^2y}{dx^2} = 9Ae^{3x} + 9Be^{-3x}$

6. If $y = Ae^{x} + Be^{-x}$, then Prove that $\frac{d^2y}{dx^2} - y = 0$

Solution:

$y = Ae^{x} + Be^{-x} \rightarrow 1$

$\frac{dy}{dx} = Ae^{3x} - Be^{-x}$

$\frac{d^2y}{dx^2}$ = A$e^x + Be^{-x}$

$\frac{d^2y}{dx^2}$ = A$e^x + Be^{-x}$

$\frac{d^2y}{dx^2}$ = y (by 1)

Hence, $\frac{d^2y}{dx^2}$ - y = 0

7. If y = a cos nx + b sin nx, then prove that $\frac{d^2y}{dx^2}$ + n^2y = 0

Solution:

y = a cos nx + b sin nx → 1

$\frac{dy}{dx}$ = -an sin nx + bn cos nx

$\frac{d^2y}{dx^2}$ = -an^2cos nx – bn^2 sin nx

$\frac{d^2y}{dx^2}$ = -an^2cos nx – bn^2 sin nx

= -n^2 (acos nx + bsin nx)

$\frac{d^2y}{dx^2}$ = -n^2y (by 1)

Hence, $\frac{d^2y}{dx^2}$ + n^2y = 0

Part – C

1. if y = $x^2 \sin x$, then prove that $x^2y_2 - 4xy_1 + (x^2+6)y = 0$

Solution:

y = $x^2 \sin x$

y_1= $x^2 \cos x$ + sin x(2x)

y_1= $x^2 \cos x$ + 2x sin x

y_2= $-x^2 \sin x + \cos x\,(2x) + \sin x(2) + 2x \cos x$

y_2= $-x^2 \sin x + 2 \sin x + 4x \cos x$

LHS = $x^2y_2 - 4xy_1 + (x^2+6)y$

= $x^2(-x^2 \sin x + 2 \sin x + 4x \cos x) - 4x(x^2 \cos x$ + 2x sin x) + $(x^2+6)x^2 \sin x$

= $-x^4 \sin x + 2x^2 \sin x + 4x^3 \cos x - 4x^3 \cos x - 8x^2 \sin x + x^4 \sin x + 6x^2 \sin x$

= 0 = RHS

2. if y = $x^2 \cos x$, then prove that $x^2y_2 - 4xy_1 + (x^2+6)y = 0$

Solution:

y = $x^2 \cos x$

y_1= $-x^2 \sin x$ + cos x(2x)

$y_1 = -x^2 \sin x$ +2x cos x

$y_2 = -x^2 \cos x + \sin x(-2x) + 2x(-\sin x) + \cos x(2)$

$= -x^2 \cos x - 2\text{ x} \sin x\ -2x\ sin\ x + 2\cos x$

$y_2 = -x^2 \cos x - 4\text{x} \sin x\ + 2\cos x$

LHS $x^2 y_2 - 4xy_1 + (x^2+6)y$

$= x^2(-x^2 \cos x + 2\cos x - 4x \sin x) - 4x(-x^2 \sin x + 2\text{x} \cos \text{x}) + (x^2+6)x^2 \cos x$

$= -x^4 \cos x + 2x^2 \cos x - 4x^3 \sin x + 4x^3 \sin x - 8x^2 \cos x + x^4 \cos x + 6x^2 \cos x$

= 0 = RHS

3. If y = $e^x \sin x$, then prove that $y_2 - 2y_1 + 2y = 0$

Solution:

y = $e^x \sin x$

$y_1 = e^x \cos x$ + sin x (e^x)

$y_2 = -e^x \sin x + \cos x\ (e^x) + \sin x(\ e^x) + e^x \cos x$

$y_2 = 2e^x \cos x$

LHS = $y_2 - 2y_1 + 2y$

= $2e^x \cos x - 2\ (e^x \cos x$ + e^x sin x) + $2e^x \sin x$

= $2e^x \cos x - 2e^x \cos x$ - $2e^x$ sin x + $2e^x \sin x$

= 0 = RHS

4. If y = $\frac{\cos x}{x}$, then prove that $xy_2 + 2y_1 + xy = 0$

Solution:

y = $\frac{\cos x}{x}$

xy = cos x

on differentiating w.r.t 'x',

$xy_1 + y = -\sin x$

again differentiating,

$xy_2 + y_1 + y_1 = -\cos x$

$xy_2 + 2y_1 = -\cos x$

$xy_2 + 2y_1 + \cos x = 0$

$\therefore xy_2 + 2y_1 + xy = 0$

5. If y = $a\cos(\log x) + b\sin(\log x)$, then prove that $x^2 y_2 + xy_1 + y = 0$

Solution:

y = $a\cos(\log x) + b\sin(\log x)$

on differentiating w.r.t 'x',

$y_1 = \frac{-a\sin(\log x)}{x} + \frac{b\cos(\log x)}{x}$

xy_1 = - a sin(log x) + b cos (log x)

again differentiating,

$xy_2 + y_1 = \frac{-a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x}$

$x(xy_2 + y_1)$ = -a cos(log x) – b sin (log x)

$x^2y_2 + xy_1 = -y$

$\therefore x^2y_2 + xy_1 + y = 0$

Exercise

PART-A

Find $\frac{dy}{dx}$ if

1) $y = (6x + 5)^3$ 2) $y = \sqrt{\cos x}$ 3) $y = \tan^4 x$ 4) $y = e^{\sin x}$

5) y = log (sin x) 6) y = cosec mx 7) $y = \cot\sqrt{x}$ 8) y = sec (2 – 3x)

PART-B

1. If $y = e^{-3x}$, then find $\frac{d^2y}{dx^2}$ or y_2 or y'' 2. If y = sin 5x, then find $\frac{d^2y}{dx^2}$ or y_2 or y''

3. If y = sec x , then find $\frac{d^2y}{dx^2}$ or y_2 or y'' 4. If $y = e^x \cos x$, then find $\frac{d^2y}{dx^2}$ or y_2 or y''

UNIT IV PARTIAL DIFFERENTIATION AND INTEGRATION

4.1 PARTIAL DIFFERENTIATION

Functions of two or more variables:

In many applications, we come across function involving more than one independent variable. For example, Area of the rectangle depends on its length and breadth, volume of cuboids depends on its length, breadth and height.

Hence one variable u depends on more than one variable.

i.e. u = f (x, y) and v = ϕ(x, y, z)

Definition of partial differentiation:

Let u = f (x, y), then partial differentiation of u w.r.t x is defined as differentiation of u w.r.t x treating y as constant and is denoted by $\frac{\partial u}{\partial x}$

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

Similarly, partial differentiation of u w.r.t y is defined as differentiation of u w.r.t y treating x as constant and is denoted by $\frac{\partial u}{\partial y}$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Second order partial derivatives:

In General, $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are functions of x and y. They can be further differentiated partially w.r.t x and y as follows.

i. $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial^2 u}{\partial x^2}\right)$

ii. $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \left(\frac{\partial^2 u}{\partial x \partial y}\right)$

iii. $\frac{\partial}{\partial y}\left(\frac{\partial u}{y}\right) = \left(\frac{\partial^2 u}{\partial y^2}\right)$

iv. $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial^2 u}{\partial y \partial x}\right)$

WORKED EXAMPLES
PART – A

1. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ for the following:

(i) u = x + y (ii) u = $x^3 + y^3$ (iii) u = e^{x+y} (iv) u = $e^{2x + 3y}$

Solution:

i) u = x + y

$$\frac{\partial u}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = 1$$

ii) u = $x^3 + y^3$

$$\frac{\partial u}{\partial x} = 3x^2$$

$$\frac{\partial u}{\partial y} = 3y^2$$

iii) $u = e^{x+y}$

$\frac{\partial u}{\partial x} = e^{x+y}$

$\frac{\partial u}{\partial y} = e^{x+y}$

iv) $u = e^{2x+3y}$

$\frac{\partial u}{\partial x} = 2e^{2x+3y}$

$\frac{\partial u}{\partial y} = 3e^{2x+3y}$

2. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ for the following:
(i) $u = x^3 + 4x^2y + 5xy^2 + y^3$ (ii) $u = 5 \sin x + 4 \tan y$ (iii) $u = y^2\sec x$ (iv) $u = \sin 4x \cos 2y$
Solution :

i) $u = x^3 + 4x^2y + 5xy^2 + y^3$

$\frac{\partial u}{\partial x} = 3x^2 + 8xy + 5y^2$

$\frac{\partial u}{\partial y} = 4x^2+10xy+3y^2$

ii) $u = 5 \sin x + 4 \tan y$

$\frac{\partial u}{\partial x} = 5\cos x$

$\frac{\partial u}{\partial y} = 4\sec^2 y$

iii) $u = y^2\sec x$

$\frac{\partial u}{\partial x} = y^2(\sec x \tan x)$

$\frac{\partial u}{\partial y} = 2y \sec x$

iv) $u = \sin 4x \cos 2y$

$\frac{\partial u}{\partial x} = \cos 2y \cos 4x(4)$

$\therefore \frac{\partial u}{\partial x} = 4\cos 4x \cos 2y$

$\frac{\partial u}{\partial y} = \sin 4x(- \sin 2y)(2)$

$\therefore \frac{\partial u}{\partial y} = -2\sin 4x \sin 2y$

3. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ for the following:
i) $\sin (ax + by)$ ii) $\cos (ax + by)$ iii) $\tan (ax + by)$ iv) $x^3 + 5x^2y + y^3$
Solution:
i) $u = \sin (ax + by)$

$\frac{\partial u}{\partial x} = a \cos (ax + by)$

$\frac{\partial u}{\partial y}$ = b cos (ax + by)

ii) u = cos (ax + by)

$\frac{\partial u}{\partial x}$ = - a sin (ax + by)

$\frac{\partial u}{\partial y}$ = - b sin (ax + by)

iii) u = tan (ax + by)

$\frac{\partial u}{\partial x}$ = a $\sec^2$(ax + by)

$\frac{\partial u}{\partial y}$ = b $\sec^2$(ax + by)

iv) u = $x^3 + 5x^2y + y^3$

$\frac{\partial u}{\partial x} = 3x^2 + 10xy$

$\frac{\partial u}{\partial y} = 5x^2 + 3y^2$

PART- B

1. Find $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ for the following:
(i) u = $x^3 + y^3$ (ii) u = x^3tan y iii) $x^3 + 3x^2y + 3xy^2 + y^3$ iv) x^2e^{3y}
Solution:

i) u = $x^3 + y^3$

$\frac{\partial u}{\partial x} = 3x^2$

$\frac{\partial^2 u}{\partial x^2} = 6x$

$\frac{\partial u}{\partial y} = 3y^2$

$\frac{\partial^2 u}{\partial y^2} = 6y$

ii) u = x^3tan y

$\frac{\partial u}{\partial x} = 3x^2$tan y

$\frac{\partial^2 u}{\partial x^2}$ = 6x tan y

$\frac{\partial u}{\partial y} = x^3\sec^2 y$

$\frac{\partial^2 u}{\partial y^2} = x^3(2\sec y.\sec y \tan y)$

$\therefore \frac{\partial^2 u}{\partial y^2} = 2x^3\sec^2 y.\tan y$

iii) $u = x^3 + 3x^2y + 3xy^2 + y^3$

$\frac{\partial u}{\partial x} = 3x^2 + 6xy + 3y^2$

$\frac{\partial^2 u}{\partial x^2} = 6x + 6y$

$\frac{\partial u}{\partial y} = 3x^2 + 6xy + 3y^2$

$\frac{\partial^2 u}{\partial y^2} = 6x + 6y$

iv) $u = x^2 e^{3y}$

$\frac{\partial u}{\partial x} = 2x\, e^{3y}$

$\frac{\partial^2 u}{\partial x^2} = 2e^{3y}$

$\frac{\partial u}{\partial y} = 3x^2\, e^{3y}$

$\frac{\partial^2 u}{\partial y^2} = 9x^2\, e^{3y}$

2. Find $\frac{\partial^2 u}{\partial x \partial y}$ and $\frac{\partial^2 u}{\partial y \partial x}$ if $u = x^2e^{5y}$

Solution:

$u = x^2e^{5y}$

$\frac{\partial u}{\partial x} = 2xe^{5y}$

$\frac{\partial^2 u}{\partial x \partial y} = 10xe^{5y}$

$\frac{\partial u}{\partial y} = 5x^2e^{5y}$

$\frac{\partial^2 u}{\partial y \partial x} = 10xe^{5y}$

PART-C

1. If $u = x^3 + 3x^2y + 3xy^2 + y^3$, then Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u$

Solution:

$u = x^3 + 3x^2y + 3xy^2 + y^3$

$\frac{\partial u}{\partial x} = 3x^2 + 6xy + 3y^2$

$\frac{\partial u}{\partial y} = 3x^2 + 6xy + 3y^2$

LHS = $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$= x(3x^2 + 6xy + 3y^2) + y(3x^2 + 6xy + 3y^2)$

$= 3x^3 + 6x^2y + 3y^2x + 3x^2y + 6xy^2 + 3y^3$

$= 3x^3 + 9x^2y + 9xy^2 + 3y^3$

$= 3\,(x^3 + 3x^2y + 3xy^2 + y^3)$

$= 3\,u =$ **RHS**

2. If $u = x^4 + 4x^3y + 3x^2y + y^4$, then find $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2}$

Solution:

$u = x^4 + 4x^3y + 3x^2y + y^4$

$\frac{\partial u}{\partial x} = 4x^3 + 12x^2y + 6xy$

$\frac{\partial^2 u}{\partial x^2} = 12x^2 + 24xy + 6y$

$\frac{\partial u}{\partial y} = 4x^3 + 3x^2 + 4y^3$

$\frac{\partial^2 u}{\partial y^2} = 12y^2$

$\therefore x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} = x^2(12x^2 + 24xy + 6y) + y^2\,(12y^2)$

$= 12x^4 + 24x^3y + 6x^2y + 12\,y^4$

3. If $u = \log(x^2 + y^2)$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2$

Solution:

$u = \log(x^2 + y^2)$

$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2}(2x)$

$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2}(2y)$

LHS = $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$= x \frac{1}{x^2+y^2}(2x) + y \frac{1}{x^2+y^2}(2y)$

$= \frac{2x^2}{x^2+y^2} + \frac{2y^2}{x^2+y^2}$

$= \frac{2x^2+2y^2}{x^2+y^2}$

$= \frac{2(x^2+y^2)}{x^2+y^2}$

= 2 = **RHS**

4. If u = log ($x^3 + y^3$), then find $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$

Solution:

u = log ($x^3 + y^3$)

$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3}(3x^2)$

$\frac{\partial^2 u}{\partial x^2} = \frac{6x(x^3+y^3)-3x^2(3x^2)}{(x^3+y^3)^2}$

$\frac{\partial^2 u}{\partial x^2} = \frac{6x(x^3+y^3)-9x^4}{(x^3+y^3)^2}$

$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3}(3y^2)$

$\frac{\partial^2 u}{\partial x^2} = \frac{6y(x^3+y^3)-3y^2(3y^2)}{(x^3+y^3)^2}$

$\frac{\partial^2 u}{\partial x^2} = \frac{6y(x^3+y^3)-9y^4}{(x^3+y^3)^2}$

5. If u = $\frac{x}{y^2} - \frac{y}{x^2}$, then find $\frac{\partial^2 u}{\partial x \partial y}$

Solution:

u = $\frac{x}{y^2} - \frac{y}{x^2}$

$u = xy^{-2} - yx^{-2}$

$\frac{\partial u}{\partial y} = x(-2y^{-3}) - x^{-2} = -2xy^{-3} - x^{-2}$

$\frac{\partial^2 u}{\partial x \partial y} = -2y^{-3} - (-2x^{-3})$

$\therefore \frac{\partial^2 u}{\partial x \partial y} = -2y^{-3} + 2x^{-3}$

6. If u = x sin y + y sin x, then Prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Solution:

u = x sin y + y sin x

$\frac{\partial u}{\partial y}$ = x cos y + sin x

$\frac{\partial^2 u}{\partial x \partial y}$ = cos y + cos x → 1

$\frac{\partial u}{\partial x}$ = sin y + y cos x

$\frac{\partial^2 u}{\partial y \partial x}$ = cos y + cos x → 2

∴ $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ (from 1 and 2)

7. If u = $\sqrt{x^2 + y^2}$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$

Solution:

u = $\sqrt{x^2 + y^2}$

$\frac{\partial u}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$

$\frac{\partial u}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$

LHS = $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

= $x\frac{x}{\sqrt{x^2+y^2}} + y\frac{y}{\sqrt{x^2+y^2}}$

= $\frac{x^2}{\sqrt{x^2+y^2}} + \frac{y^2}{\sqrt{x^2+y^2}}$

= $\frac{(x^2+y^2)}{\sqrt{x^2+y^2}}$

= $\frac{\sqrt{x^2+y^2}.\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$

= $\sqrt{x^2 + y^2}$

= u = **RHS**

EXERCISE

PART-A

1. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, if u = $3x^3 + 6x^2y + 2xy^2 + 3y^3$

2. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, if u = 5 cos x + 4 sec y
3. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, if u = y^2tan x
4. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, if u = cot 4x cosec 2y

PART-B

1. Find $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$, if u = $2x^3 + 2y^3$
2. Find $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$, if u = x^3sin y
3. Find $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$, if u = $3x^3 + 6x^2y + 6xy^2 + 3y^3$
4. Find $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$, if u = x^2e^{4y}

PART-C

1. If u = x cos y + y cos x , then Prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$
2. If u = $x^4 + 4x^3y^2 + 3x^2y^2 + y^4$, then find $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2}$

4.2 INTEGRATION

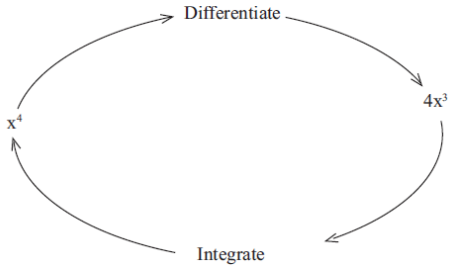
Sir Sardar Vallabhai Patel, called the Iron Man of India integrated several princely states together while forming our country Indian Nation after independence. Like that in Mathematics while finding area under a curve through integration, the area under the curve is divided into smaller rectangles and then integrating (i.e) summing of all the area of rectangles together. So, integration means of summation of very minute things of the same kind.

Integration as the reverse of differentiation:

Integration can also be introduced in another way, called integration as the reverse of differentiation.

Differentiation in reverse:

Suppose we differentiate the function y = x^4, then we have $\frac{dy}{dx} = 4x^3$. Now, we say that integral of $4x^3$ is x^4 and we write this as $\int 4x^3dx = x^4$. Pictorially, we can think of this as follows:



Suppose we differentiate the functions, $y = x^4 + 5$, $y = x^4 - 23$, $y = x^4 + 100$, then also we get $\frac{dy}{dx} = 4x^3$. Now, what could we say about the integral of $4x^3$. Can we say that it is $x^4 + 5$ (or) $x^4 - 23$ (or) $x^4 + 100$? So, in general we say that $\int 4x^3 dx = x^4 + c$ where c is called constant of integration.

The symbol for integration is $\int$, known as integral sign. Along with the integral sign there is a term dx which must always be written and which indicates the name of the variable involved, in this case 'x'. Technically integrals of this sort are called indefinite Integrals.

List of Formulae:

S.No.	Integration	Reverse Process of Differentiation
1.	$\int x^n dx = \frac{x^{n+1}}{n+1} + c\ ;\ (n \neq -1)$	$\frac{d}{dx}\left[\frac{x^{n+1}}{n+1} + c\right] = (\cancel{n+1})\frac{x^{n+1-1}}{\cancel{n+1}} + 0$ $= x^n$
2.	$\int \frac{1}{x} dx = \log x + c$	$\frac{d}{dx}(\log x + c) = \frac{1}{x} + 0 = \frac{1}{x}$
3.	$\int e^x dx = e^x + c$	$\frac{d}{dx}(e^x + c) = e^x$
4	$\int \sin x\, dx = -\cos x + c$	$\frac{d}{dx}(-\cos x + c) = \sin x$
5.	$\int \cos x\, dx = \sin x + c$	$\frac{d}{dx}(\sin x + c) = \cos x$
6.	$\int \sec^2 x dx = \tan x + c$	$\frac{d}{dx}(\tan x + c) = \sec^2 x$
7.	$\int \mathrm{cosec}^2 x dx = -\cot x + c$	$\frac{d}{dx}(-\cot x + c) = \mathrm{cosec}^2 x$
8.	$\int \sec x \tan x\, dx = \sec x + c$	$\frac{d}{dx}(\sec x + c) = \sec x \tan x$
9.	$\int \mathrm{cosec}\, x \cot x\, dx = -\mathrm{cosec}\, x + c$	$\frac{d}{dx}(-\mathrm{cosec}\, x + c) = \mathrm{cosec}\, x \cot x$

Note

Properties of integration

1. $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)\, dx$
2. $\int cf(x)dx = c\int f(x)dx$ where c is any constant

Integration of the function f(ax+b)

1. $\int (ax+b)^n dx = \frac{1}{a}\left[\frac{(ax+b)^{n+1}}{n+1}\right] + c$

2. $\int \frac{1}{ax+b} dx = \frac{1}{a}\log(ax+b) + c$

3. $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c$

4. $\int \sin(ax+b)dx = \frac{-1}{a}\cos(ax+b) + c$

5. $\int cos(ax+b) = \frac{1}{a}\sin(ax+b) + c$

6. $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + c$

7. $\int \mathrm{cosec}^2(ax+b)dx = \frac{-1}{a}\cot(ax+b) + c$

8. $\int \sec(ax+b)\tan(ax+b)dx = \frac{1}{a}\sec(ax+b) + c$

9. $\int \mathrm{cosec}(ax+b)\cot(ax+b)dx = \frac{-1}{a} co\sec(ax+b) + c$

WORKED PROBLEMS

PART – A

1. Evaluate: $\int \mathrm{xdx}$

   Solution:

   $$\int \mathrm{x\,dx} = \frac{\mathrm{x}^{1+1}}{1+1} + \mathrm{c}$$

   $$= \frac{\mathrm{x}^2}{2} + \mathrm{c}$$

2. Evaluate: $\int \mathrm{dx}$

   Solution:

   $$\int \mathrm{dx} = \mathrm{x} + \mathrm{c}$$

3. Evaluate: $\int \frac{1}{\mathrm{x}^4}\mathrm{dx}$

   Solution:

   $$\int \frac{1}{\mathrm{x}^4}\mathrm{dx} = \mathrm{x}^{-4} + \mathrm{c}$$

   $$= \frac{\mathrm{x}^{-4+1}}{-4+1} + \mathrm{c}$$

   $$= \frac{\mathrm{x}^{-3}}{-3} + \mathrm{c}$$

4. Evaluate: $\int(\sqrt{x}+\frac{1}{\sqrt{x}})dx$

Solution:

$$\int(\sqrt{x}+\frac{1}{\sqrt{x}})dx = \int x^{\frac{1}{2}}dx + \int x^{-\frac{1}{2}}dx$$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \frac{2x^{\frac{3}{2}}}{3} + 2\sqrt{x} + c$$

5. Evaluate: $\int(x^2-3x)dx$

Solution:

$$\int(x^2-3x)dx = \int x^2dx - 3\int xdx$$

$$= \frac{x^{2+1}}{2+1} - 3\frac{x^{1+1}}{1+1} + c$$

$$= \frac{x^3}{3} - 3\frac{x^2}{2} + c$$

6. Evaluate: $\int(e^{3x}+e^{-5x})dx$

Solution:

$$\int(e^{3x}+e^{-5x})dx$$

$$= \frac{e^{3x}}{3} + \frac{e^{-5x}}{-5} + c$$

$$= \frac{e^{3x}}{3} - \frac{e^{-5x}}{5} + c$$

7. Evaluate: $\int \sec^2(4x)dx$

Solution:

$$\int \sec^2(4x)dx = \frac{1}{4}\tan 4x + c$$

8. Evaluate: $\int(x^2+x+4)dx$

Solution:

$$\int(x^2+x+4)dx = \int x^2dx + \int xdx + \int 4\,dx$$

$$= \frac{x^{2+1}}{2+1} + \frac{x^{1+1}}{1+1} + 4x + c$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 4x + c$$

9. Evaluate: $\int \tan^2 x \, dx$

Solution:

$$= \int \tan^2 x \, dx$$

$$= \int (sec^2 x - 1) dx$$

$$= \tan x - x + c$$

PART – B

1. Evaluate: $\int (x+1)(x+2) dx$

Solution:

$$\int (x+1)(x+2) dx = \int (x^2 + 2x + x + 2) dx$$

$$= \int (x^2 + 3x + 2) dx$$

$$= \int x^2 dx + 3 \int x dx + 2 \int dx$$

$$= \frac{x^{2+1}}{2+1} + 3\frac{x^{1+1}}{1+1} + 2x + c$$

$$= \frac{x^3}{3} + 3\frac{x^2}{2} + 2x + c$$

$$= \frac{x^3}{3} + \frac{3x^2}{2} + 2x + c$$

2. Evaluate: $\int (x+1)(2x-3) dx$

Solution:

$$\int (x+1)(2x-3) dx = \int (2x^2 - 3x + 2x - 3) dx$$

$$= \int (2x^2 - x - 3) dx$$

$$= \frac{2x^3}{3} - \frac{x^2}{2} - 3x + c$$

3. Evaluate: $\int \left(x^3 + e^x + \frac{1}{x}\right) dx$

Solution:

$$\int \left(x^3 + e^x + \frac{1}{x}\right) dx = \int x^3 dx + \int e^x dx + \int \frac{dx}{x}$$

$$= \frac{x^4}{4} + e^x + \log x + c$$

4. Evaluate: $\int(x^3 - 5\sec^2 x + \frac{7}{x})dx$

Solution:

$$\int(x^3 - 5\sec^2 x + \frac{7}{x})dx = \int x^3 dx - 5\int \sec^2 x\, dx + 7\int \frac{1}{x} dx$$

$$= \frac{x^4}{4} - 5\tan x + 7\log x + c$$

5. Evaluate: $\int(e^x + \frac{1}{x} + \sec x \tan x)dx$

Solution:

$$\int\left(e^x + \frac{1}{x} + \sec x \tan x\right) dx = \int e^x dx + \int \frac{1}{x} dx + \int \sec x \tan x\, dx$$

$$= e^x + \log x + \sec x + c$$

6. Evaluate: $\int \frac{x^3+2x^2-3x}{x^2} dx$

Solution:

$$\int \frac{x^3+2x^2-3x}{x^2} dx = \int \frac{x^3}{x^2} dx + \int \frac{2x^2}{x^2} dx - 3\int \frac{x}{x^2} dx$$

$$= \int x\, dx + \int 2\, dx - 3\int \frac{1}{x} dx$$

$$= \frac{x^2}{2} + 2x - 3\log x + c$$

7. Evaluate: $\int \frac{3x^2+2x+7}{x} dx$

Solution:

$$\int \frac{3x^2+2x+7}{x} dx = \int \frac{3x^2}{x} dx + \int \frac{2x}{x} dx + 7\int \frac{1}{x} dx$$

$$= \int 3x\, dx + \int 2\, dx + 7\int \frac{1}{x} dx$$

$$= \frac{3x^2}{2} + 2x + 7\log x + c$$

8. Evaluate: $\int(x^2 + x + 1)(x^2 - x + 1)dx$

Solution:

$$\int(x^2 + x + 1)(x^2 - x + 1)dx = \int(x^4 - x^3 + x^2 + x^3 - x^2 + x + x^2 - x + 1)dx$$

$$= \int(x^4 + x^2 + 1)dx$$

$$= \frac{x^5}{5} + \frac{x^3}{3} + x + c$$

9. Evaluate: $\int \sec x(\sec x - \tan x)\, dx$

Solution:

$\int \sec x(\sec x - \tan x)\, dx = \int (sec^2 x - \sec x \tan x) dx$

$= \tan x - \sec x + c$

10. Evaluate: $\int \sqrt{1 - \sin 2x}\, dx$

Solution:

$$\int \sqrt{1 - \sin 2x}\ dx = \int \sqrt{\sin^2 x + \cos^2 x - 2 \sin x \cos x}\ dx$$

$$= \int \sqrt{(\sin x - \cos x)^2}\ dx$$

$$= \int (\sin x - \cos x)\ dx$$

$$= -\cos x - \sin x + c$$

11. Evaluate: $\int \frac{\cos^2 x}{1+\sin\ x} dx$

Solution:

$$\int \frac{\cos^2 x}{1+\sin\ x} dx = \int \frac{\cos^2 x}{1+\sin\ x} \times \frac{1-\sin x}{1-\sin x} dx \quad [(a+b)(a-b) = a^2 - b^2]$$

$$= \int \frac{\cos^2 x(1-\sin x)}{1-\sin^2 x} dx$$

$$= \int \frac{\cos^2 x(1-\sin x)}{\cos^2 x} dx$$

$$= \int (1 - \sin x)\ dx$$

$$= x + \cos x + c$$

12. Evaluate: $\int \frac{\sin^2 x}{1-\cos\ x} dx$

Solution:

$$\int \frac{\sin^2 x}{1-\cos\ x} dx = \int \frac{\sin^2 x}{1-\cos\ x} \times \frac{1+\cos x}{1+\cos x} dx \quad [(a+b)(a-b) = a^2 - b^2]$$

$$= \int \frac{\sin^2 x(1+\cos x)}{1-\cos^2 x} dx$$

$$= \int \frac{\sin^2 x(1+\cos x)}{\sin^2 x} dx$$

$$= \int (1 + \cos x)\ dx$$

$$= x + \sin x + c$$

13. Evaluate: $\int(\tan x + \cot x)^2 dx$

Solution:

$\int(\tan x + \cot x)^2 dx = \int(tan^2 x + cot^2 x + 2\tan x \cot x)dx$

$= \int(sec^2 x - 1 + cosec^2 x - 1 + 2\tan x \ \frac{1}{\tan x})dx$

$= \int(sec^2 x + cosec^2 x)dx$

= tan x – cot x + c

14. Evaluate: $\int \sin^2 x \, dx$

Solution:

$\int \sin^2 x \, dx$ $\qquad$ $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$= \frac{1}{2}\int (1 - \cos 2x)dx$

$= \frac{1}{2}(x - \frac{\sin 2x}{2}) + c$

15. Evaluate: $\int \sin^3 x \, dx$

Solution:

$\int \sin^3 x \, dx$ $\qquad$ $\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$

$= \int \frac{1}{4}(3\sin x - \sin 3x)dx$

$= \frac{1}{4}[-3\cos x + \frac{cos3x}{3}] + c$

16. Evaluate: $\int \sin 5x \cos 3x \, dx$

Solution:

$\int \sin 5x \cos 3x \, dx$

$= \frac{1}{2}\int [\sin(5x + 3x) + \sin(5x - 3x)] \, dx$ $\quad$ [sin(A+B) + sin (A-B) = 2sin A cos B]

$= \frac{1}{2}\int(\sin 8x + \sin 2x) \, dx$

$= \frac{1}{2}\left[-\frac{\cos 8x}{8} - \frac{\cos 2x}{2}\right] + c$

$= -\frac{\cos 8x}{16} - \frac{\cos 2x}{4} + c$

17. Evaluate: $\int \sin 7x \sin 4x \, dx$

Solution:

$\int \sin 7x \sin 4x \, dx$

$= \frac{-1}{2}\int [\cos(7x + 4x) - \cos(7x - 4x)] \, dx$ [cos(A+B)-cos(A-B) = -2sinAsinB]

$= \frac{-1}{2}\int (\cos 11x - \cos 3x) \, dx$

$$= \frac{-1}{2}\left[\frac{\sin 11x}{11} - \frac{\sin 3x}{3}\right] + c$$

$$= \frac{-\sin 11x}{22} + \frac{\sin 3x}{6} + c$$

18. Evaluate: $\int (\frac{2}{x^2} - \frac{7}{x} + \frac{3}{sin^2 x})dx$

Solution:

$$= \int (\frac{2}{x^2} - \frac{7}{x} + \frac{3}{sin^2 x})dx$$

$$= \int (2x^{-2} - \frac{7}{x} + 3cosec^2 x)\, dx$$

$$= -2\frac{x^{-3}}{3} - 7log\ x - 3\cot x + c$$

EXERCISE

PART-A

1. Evaluate: $\int (e^{-6x} + e^{-5x})dx$
2. Evaluate: $\int \sin 3x dx$
3. Evaluate: $\int (3x^3 + 2x + 4)dx$
4. Evaluate: $\int (x^3 + \text{cosec}^2 x + \frac{1}{x})dx$
5. Evaluate: $\int \cot^2 x\, dx$

PART- B

1. Evaluate: $\int (3x^3 - 5\sec x \cot x + \frac{4}{x})dx$
2. Evaluate: $\int (x^2 + 2x + 1)\,(3x^2 + x + 3)dx$
3. Evaluate: $\int (x - 1)(2x + 3)dx$
4. Evaluate: $\int (3x - 2)(x - 2)dx$
5. Evaluate: $\int \sqrt{1 + \sin 2x}\, dx$
6. Evaluate: $\int \frac{\cos^2 x}{1 - \sin x} dx$
7. Evaluate: $\int \frac{\sin^2 x}{1 + \cos x} dx$
8. Evaluate: $\int \cos^2 x\, dx$
9. Evaluate: $\int \cos^3 x\, dx$

4.3 Integration by substitution

Rule I: $\int \frac{f'(x)}{f(x)} dx = \log f(x) + c$

Rule II: $\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+`1}}{n+1} + c$

RuleIII: $\int F[f(x)]f'(x)dx = \int F(u)du + c$, after integrating replace u by f(x)

WORKED EXAMPLE
PART – A

1. Evaluate : $\int \frac{2x}{1+x^2} dx$

   Solution:

   $$\int \frac{2x}{1+x^2} dx = \log(1 + x^2) + c$$

2. Evaluate : $\int \frac{e^x}{e^x-1} dx$

   Solution:

   $$\int \frac{e^x}{e^x-1} dx = \log(e^x - 1) + c$$

3. Evaluate : $\int \frac{2x-7}{x^2-7x+9} dx$

   Solution:

   $$\int \frac{2x-7}{x^2-7x+9} dx = \log(x^2 - 7x + 9) + c$$

4. Evaluate : $\int \frac{1+\cos x}{x+\sin x} dx$

   Solution:

   $$\int \frac{1+\cos x}{x+\sin x} dx = \log(x + \sin x) + c$$

5. Evaluate : $\int \frac{e^x-e^{-x}}{e^x+e^{-x}} dx$

   Solution:

   $$\int \frac{e^x-e^{-x}}{e^x+e^{-x}} dx = \log(e^x + e^{-x}) + c$$

6. Evaluate : $\int \frac{\sin x}{1-\cos x} dx$

   Solution:

   $$\int \frac{\sin x}{1-\cos x} dx = \log(1 - \cos x) + c$$

7. Evaluate : $\int \frac{\sec^2 x}{4+\tan x} dx$

Solution:

$$\int \frac{\sec^2 x}{4+\tan x} dx = \log(4 + \tan x) + c$$

8. Evaluate: $\int [x^2 - 5x + 9]^{11}(2x - 5)dx$

Solution:

$$\int [x^2 - 5x + 9]^{11}(2x - 5)dx$$

$$= \frac{[x^2-5x+9]^{11+`1}}{11+1} + c$$

$$= \frac{[x^2-5x+9]^{12}}{12} + c$$

PART-B

1. Evaluate: $\int \frac{x}{x^2+1} dx$

Solution:

$\int \frac{x}{x^2+1} dx$

$$= \frac{1}{2} \int \frac{2x}{x^2+1} dx$$

$$= \frac{1}{2} \log(x^2 + 1) + c$$

2. Evaluate: $\int \cot x \, dx$

Solution:

$\int \cot x \, dx$

$$= \int \frac{\cos x}{\sin x} dx$$

$$= \log (\sin x) + c$$

3. Evaluate: $\int \tan x \, dx$

Solution:

$\int \tan x \, dx$

$$= \int \frac{\tan x \sec x}{\sec x} dx$$

$$= \log (\sec x) + c$$

4. Evaluate : $\int \frac{1}{x \log x} dx$

Solution:

$\int \frac{1}{x \log x} dx$

$$= \int \frac{\frac{1}{x}}{\log x} dx$$

$$= \log(\log x) + c$$

5. Evaluate : $\int \frac{\cot x}{\log (\sin x)} dx$

Solution:

$\int \frac{\cot x}{\log (\sin x)} dx$

$$= \int \frac{\frac{\cos x}{\sin x}}{\log (\sin x)} dx$$

$$= \log(\log (\sin x)) + c$$

6. Evaluate: $\int \sec x \, dx$

Solution:

$\int \sec x \, dx \quad = \int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x} dx$

$$= \int \frac{\sec^2 x+\sec x \tan x}{\sec x+\tan x} dx$$

$$= \log(\sec x + \tan x) + c$$

7. Evaluate: $\int \operatorname{cosec} x \, dx$

Solution:

$\int \operatorname{cosec} x \, dx \quad = \int \frac{\operatorname{cosec} x \, (\operatorname{cosec} x-\cot x)}{\operatorname{cosec} x-\cot x} dx$

$$= \int \frac{\operatorname{cosec}^2 x-\operatorname{cosec} x \cot x}{\operatorname{cosec} x-\cot x} dx$$

$$= \log(\operatorname{cosec} x - \cot x) + c$$

8. Evaluate: $\int \frac{(\log x)^5}{x} dx$

Solution

$\int \frac{(\log x)^5}{x} dx$

$$= \int (\log x)^5 \frac{1}{x} dx$$

$$= \frac{(\log x)^6}{6} + c$$

9. Evaluate: $\int \tan^4 x \sec^2 x \, dx$

Solution:

$$\int \tan^4 x \sec^2 x \, dx = \int (\tan x)^4 sec^2 x \, dx$$

$$= \frac{\tan^5 x}{5} + c$$

10. Evaluate: $\int \sin^3 x \cos x \, dx$

Solution:

$$\int \sin^3 x \cos x \, dx = \int (\sin x)^3 \cos x \, dx$$

$$= \frac{\sin^4 x}{4} + c$$

11. Evaluate: $\int (2x^2 - 8x + 5)^{11} (x - 2) dx$

Solution:

$$= \int (2x^2 - 8x + 5)^{11} (x - 2) dx$$

$$= \int \frac{4}{4} (2x^2 - 8x + 5)^{11} (x - 2) dx$$

$$= \frac{1}{4} \int (2x^2 - 8x + 5)^{11} (4x - 8) dx$$

$$= \frac{1}{4} \frac{(2x^2 - 8x + 5)^{12}}{12} + c$$

$$= \frac{(2x^2 - 8x + 5)^{12}}{48} + c$$

12. Evaluate: $\int \sqrt{\sin x} \cos x \, dx$

Solution:

$$= \int \sqrt{\sin x} \cos x \, dx$$

$$= \int (\sin x)^{\frac{1}{2}} \cos x \, dx$$

$$= \frac{(\sin x)^{\frac{3}{2}}}{3/2} + c$$

13. Evaluate: $\int \sqrt{\tan x}\sec^2 x\, dx$

Solution:

$$= \int \sqrt{\tan x}\sec^2 x\, dx$$

$$= \int (\tan x)^{\frac{1}{2}} \sec^2 x\, dx$$

$$= \frac{(\tan x)^{\frac{3}{2}}}{3/2} + c$$

14. Evaluate: $\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx$

Solution:

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx$$

Let u = $\sqrt{x}$ $\therefore \frac{du}{dx} = \frac{1}{2\sqrt{x}}$

2du = $\frac{1}{\sqrt{x}} dx$

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx = \int 2\sin u\ du$$

$$= 2\int \sin u\ du$$

$$= -2\cos u + c \quad = -2\cos\sqrt{x} + c$$

15. Evaluate: $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx$

Solution:

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx$$

Let u = $\sqrt{x}$ $\therefore \frac{du}{dx} = \frac{1}{2\sqrt{x}}$

2du = $\frac{1}{\sqrt{x}} dx$

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx = \int 2\cos u\ du$$

$$= 2\int \cos u\ du$$

$$= 2\sin u + c \quad = -2\cos\sqrt{x} + c$$

16. Evaluate: $\int \frac{\sin(logx)}{x} dx$

Solution:

$$\int \frac{\sin(\log x)}{x} dx$$

Let u = $\log x$ $\therefore \frac{du}{dx} = \frac{1}{x}$

du = $\frac{1}{x} dx$

$\int \frac{\sin(logx)}{x} dx = \int \sin u \ du$

= - cos u + c = - cos(log x) + c

17. Evaluate: $\int \frac{\sec^2(logx)}{x} dx$

Solution:

$$\int \frac{\sec^2(\log x)}{x} dx$$

Let u = $\log x$ $\therefore \frac{du}{dx} = \frac{1}{x}$

du = $\frac{1}{x} dx$

$\int \frac{\sec^2(logx)}{x} dx = \int \sec^2 u \ du$

= tan u + c = tan(log x) + c

18. Evaluate: $\int e^{\tan x} sec^2 x \ dx$

Solution:

$$\int e^{\tan x} sec^2 x \ dx$$

Let u = tan x $\therefore \frac{du}{dx} = \sec^2 x$

du = $\sec^2 x \ dx$

$\int e^{\tan x} sec^2 x \ dx = \int e^u \ du$

= e^u + c = $e^{\tan x}$+ c

19. Evaluate: $\int e^{\sin^2 x} \sin 2x \ dx$

Solution:

$\int e^{\sin^2 x} \sin 2x \ dx$

Let u = $\sin^2 x$ ∴ $\frac{du}{dx}$ = 2 sin x cos x

du = 2 sin x cos x dx

du = sin 2x dx

$\int e^{\sin^2 x} \sin 2x \ dx = \int e^u \ du$

= e^u + c = $e^{sin^2 x}$ + c

20. Evaluate: $\int \frac{2x+3}{\sqrt{x^2+3x-4}} \ dx$

Solution:

$\int \frac{2x+3}{\sqrt{x^2+3x-4}} \ dx$

Let u = $x^2 + 3x - 4$ ∴ $\frac{du}{dx}$ = 2x + 3

du = (2x + 3) dx

$\int \frac{2x+3}{\sqrt{x^2+3x-4}} \ dx = \int \frac{1}{\sqrt{u}} du$

= $\int u^{-1/2} du$

= $\frac{u^{1/2}}{1/2} + c$ = $2\sqrt{u} + c$

= $2\sqrt{x^2 + 3x - 4} + c$

EXERCISE

PART-A

1. Evaluate : $\int \frac{e^x}{e^x+1} dx$
2. Evaluate : $\int \frac{2lx+m}{lx^2+mx+n} dx$
3. Evaluate : $\int \frac{\sin x+\cos x}{\sin x-\cos x} dx$
4. Evaluate: $\int (4x-1)(2x^2-x+5)^4 \ dx$
5. Evaluate: $\int \sin^4 x \cos x \, dx$
6. Evaluate: $\int (2x+3)\sqrt{x^2+3x-5} \, dx$

PART-B

1. Evaluate: $\int 5x^4 e^{x^5} \ dx$
2. Evaluate: $\int \frac{\cos x}{\sqrt{\sin x}} dx$
3. Evaluate: $\int x^2 sec^2(x^3) \, dx$
4. Evaluate: $\int x^2 cos^2(x^3) \, dx$

UNIT V VECTOR ALGEBRA

5.1 VECTOR INTRODUCTION

Scalar :

A scalar quantity is one which has magnitude only.

Example : Length, temperature, mass, time.

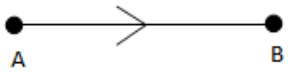
Vecetor :

A vector quantity is one which has both magnitude and direction.

Example : Force , displacement, velocity, acceleration, momentum.

<u>Representation of vector</u>

A vector is a directed line segment. The length of the segment is called magnitude of the vector. The direction is indicated by an arrow joining the initial point A and final point B of the line segment AB.



The magnitude of the vector $\overrightarrow{AB}$ is AB = $|\overrightarrow{AB}|$.

Zero vector or Null vector:

A zero vector is one whose magnitude is zero, but no definite direction associated with it. For example, if A is a point, $\overrightarrow{AA}$ is a zero vector.

Unit vector:

A vector of magnitude one unit is called an unit vector. If $\vec{a}$ is an unit vector, then it is also denoted as $\hat{a}$ .ie, $|\hat{a}| = |\vec{a}| = 1$.

Negative vector:

If $\overrightarrow{AB}$ is a vector, then the negative vector of $\overrightarrow{AB}$ is $\overrightarrow{BA}$. If the direction of a vector is changed then we get the negative vector.

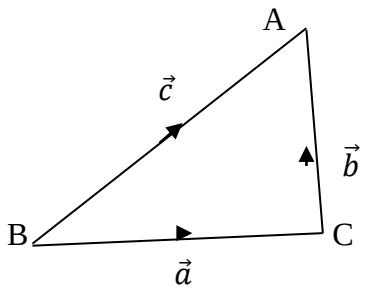
Ie, $\overrightarrow{BA} = -\overrightarrow{AB}$

Equal vectors:

Two vectors are said to be equal, if they have the same magnitude and the same direction, but it is not required to have the same segment for the two vectors.

For example, in a parallelogram ABCD, $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AD} = \overrightarrow{BC}$.

Addition of two vectors:



If $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$ and $\overrightarrow{BA} = \vec{c}$,then $\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$, ie. $\vec{a} + \vec{b} = \vec{c}$

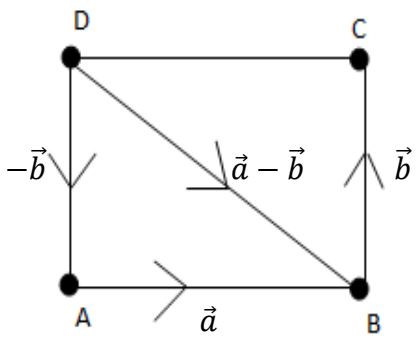
If the end point of first vector and the initial point of the second vector are same, the addition of two vectors can be formed as the vector joining the initial point of the first vector and the end point of the second vector.

Properties of vector addition:

1) Vector addition is commutative. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2) Vector addition is associative. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.

Subtraction of vector:

If $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$ $\therefore \overrightarrow{CB} = -\vec{b}$



$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

$= \overrightarrow{AB} + \overrightarrow{CB}$

$= \overrightarrow{AB} + \overrightarrow{DA}$ [$\because$ $\overrightarrow{CB}$ and $\overrightarrow{DA}$ are equal]

$= \overrightarrow{DA} + \overrightarrow{AB}$ [$\because$ addition is commutative]

$= \overrightarrow{DB}$

Multiplication by a scalar:

If $\vec{a}$ is a given vector and $\lambda\vec{a}$ is a vector whose magnitude is $\lambda|\vec{a}|$ and whose direction is the same to that of $\vec{a}$, provided λ is a positive quantity. If λ is a negative $\lambda\vec{a}$ is a vector whose magnitude is $|\lambda|\ |\vec{a}|$ and whose direction is opposite to that of $\vec{a}$.

Properties:

1) $(m+n)\vec{a} = m\vec{a} + n\vec{a}$
2) $m(n\vec{a}) = n(m\vec{a}) = mn\vec{a}$
3) $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$

Collinear vectors:

If $\vec{a}$ and $\vec{b}$ are such that they have the same or opposite direction they are said to be collinear vectors and one is numerical multiple of the other, ie. $\vec{b} = k\vec{a}$ or $\vec{a} = k\vec{b}$

Coplanar of Vectors:

Let $\vec{a}, \vec{b}, \vec{c}$ be coplanar vectors such that no two vectors are parallel. Then there exists scalars α and β such that $\vec{c} = \alpha\vec{a} + \beta\vec{b}$
Similarly, we can get constants (scalars) such that $\vec{a} = \alpha'\vec{b} + \beta\vec{c}$ and $\vec{b} = \alpha''\vec{c} + \beta''\vec{a}$

If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are four vectors, no three of which are coplanar, then there exist scalars l, m, n such that $\vec{d} = l\vec{a} + m\vec{b} + n\vec{c}$

Position vector:

If P is any point in the space and 0 is the origin , then $\overrightarrow{OP}$ is called the position vector of the point P.

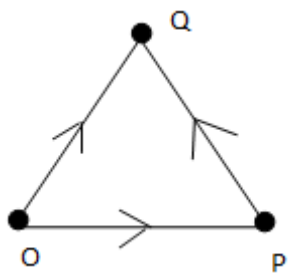
Decomposition of a vector into mutually perpendicular axes:

Let P be a point in a plane. Let O be the origin and $\vec{\imath}$ and $\vec{\jmath}$ the unit vectors along the x and y axes in that plane Then if P is (α, β), the position vector of the point P is $\overrightarrow{OP} = \alpha\vec{\imath} + \beta\vec{\jmath}$

Similarly if P is any point (x, y, z)in the space, $\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$ be the unit vectors along the x, y, z axes in the space ,the position vector of the point P is $\overrightarrow{OP} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ .The magnitude of $\overrightarrow{OP}$ is OP $=|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$

Distance between the points:

If P and Q are two points in the space with coordinates P(x , y , z) and Q(x, y , z), then the position vectors are $\overrightarrow{OP} = x_1\vec{\imath} + y_1\vec{\jmath} + z_1\vec{k}$ and $\overrightarrow{OQ} = x_2\vec{\imath} + y_2\vec{\jmath} + z_2\vec{k}$



$\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$\overrightarrow{PQ} = \vec{\imath}\ (x_2 - x_1) + \vec{\jmath}\ (y_2 - y_1) + \vec{k}\ (z_2 - z_1)$

$= |\vec{\imath}\ (x_2 - x_1) + \vec{\jmath}\ (y_2 - y_1) + \vec{k}\ (z_2 - z_1)|$

∴Distance between P and Q is PQ = $|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Direction Cosines:

Let AB be a straight line making angles α , β ,γ with the co-ordinates axes x'ox, y'oy ,z'oz respectively. Then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines of the line AB and denoted by l , m , n .Let $\overrightarrow{OP}$ be parallel to $\overrightarrow{AB}$ and P be (x, y, z).Then $\overrightarrow{OP}$ also makes angles α , β ,γ with x, y, and z axes.

Now, OP = r = $\sqrt{x^2 + y^2 + z^2}$

Then, $\cos\alpha = \frac{x}{r}$, $\cos\beta = \frac{y}{r}$, $\cos\gamma = \frac{z}{r}$

Now, sum of squares of the direction cosines of any straight line is

$l^2 + m^2 + n^2 = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 + \left(\frac{z}{r}\right)^2$

$= \frac{x^2+y^2+z^2}{r^2} = \frac{r^2}{r^2} = 1$

Direction ratios: Let $\hat{n}$ be the unit vector along $\overrightarrow{OP}$

Then, $\hat{n} = \frac{\overrightarrow{OP}}{|\overrightarrow{OP}|} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{r}$

$= \frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k} = = l\vec{i} + m\vec{j} + n\vec{k}$

Any three numbers p, q, r proportional to the direction cosines of the straight line AB are called the direction ratios of the straight line AB.

WORKED EXAMPLES
PART - A

1. If $\vec{a} = 3\vec{i} - 4\vec{j} + 6\vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} + 3\vec{k}$ find i) $3\vec{a} + 4\vec{b}$ ii) $2\vec{a} - \vec{b}$

Solution:

Given

$\vec{a} = 3\vec{i} - 4\vec{j} + 6\vec{k}$

$\vec{b} = 2\vec{i} + \vec{j} + 3\vec{k}$

$3\vec{a} = 3(3\vec{i} - 4\vec{j} + 6\vec{k}) = 9\vec{i} - 12\vec{j} + 18\vec{k}$

$4\vec{b} = 4(2\vec{i} + \vec{j} + 3\vec{k}) = 8\vec{i} + 4\vec{j} + 12\vec{k}$

$3\vec{a} + 4\vec{b} = 9\vec{i} - 12\vec{j} + 18\vec{k} + 8\vec{i} + 4\vec{j} + 12\vec{k}$

$= 17\vec{i} - 8\vec{j} + 30\vec{k}$

$2\vec{a} = 2(3\vec{i} - 4\vec{j} + 6\vec{k}) = 6\vec{i} - 8\vec{j} + 12\vec{k}$

$\vec{b} = 2\vec{i} + \vec{j} + 3\vec{k}$

$2\vec{a} - \vec{b} = (6\vec{i} - 8\vec{j} + 12\vec{k}) - (2\vec{i} + \vec{j} + 3\vec{k})$

$= 4\vec{i} - 9\vec{j} + 9\vec{k}$

2. Find the unit vector along $\vec{i} - \vec{j} + 3\vec{k}$

Solution:

Let $\vec{a} = \vec{i} - \vec{j} + 3\vec{k}$

$|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$

$= \sqrt{1^2 + (-1)^2 + 3^2}$

$= \sqrt{1 + 1 + 9}$

$= \sqrt{11}$

Unit vector along $\vec{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{i} - \vec{j} + 3\vec{k}}{\sqrt{11}}$

3. Find the unit vector along $4\vec{i} - 5\vec{j} + 7\vec{k}$

Solution:

Let $\vec{a} = 4\vec{i} - 5\vec{j} + 7\vec{k}$

$|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$

$= \sqrt{4^2 + (-5)^2 + 7^2}$

$= \sqrt{16 + 25 + 49}$

$= \sqrt{90}$

Unit vector along $\vec{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{4\vec{i} - 5\vec{j} + 7\vec{k}}{\sqrt{90}}$

4. Find the direction cosines of the vector $2\vec{i} + 3\vec{j} - 4\vec{k}$

Solution:

Let $\vec{a} = 2\vec{i} + 3\vec{j} - 4\vec{k}$

$r = |\vec{a}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{2^2 + 3^2 + (-4)^2} = \sqrt{4 + 9 + 16}$

$= \sqrt{29}$

Direction cosines of $\overrightarrow{AB}$ are

$\cos\alpha = \frac{x}{r} = \frac{2}{\sqrt{29}}$, $\cos\beta = \frac{y}{r} = \frac{3}{\sqrt{29}}$, $\cos\gamma = \frac{z}{r} = \frac{-4}{\sqrt{29}}$

5. Find the direction cosines of the vector $2\vec{i} + \vec{j} + \vec{k}$

Solution:

Let $\vec{a} = 2\vec{i} + \vec{j} + \vec{k}$

$r = |\vec{a}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{2^2 + 1^2 + (1)^2} = \sqrt{4 + 1 + 1}$

$= \sqrt{6}$

Direction cosines of $\overrightarrow{AB}$ are

$\cos\alpha = \frac{x}{r} = \frac{2}{\sqrt{6}}$, $\cos\beta = \frac{y}{r} = \frac{1}{\sqrt{6}}$, $\cos\gamma = \frac{z}{r} = \frac{1}{\sqrt{6}}$

6. Find the direction ratios of the vector $\vec{i}$ +2j -$\vec{k}$

Solution:

The direction ratios of $\vec{i}$ + 2 $\vec{j}$ - $\vec{k}$ are 1: 2: -1

7. If position vectors of the points A and B are $2\vec{i} + \vec{j} - \vec{k}$, $5\,\vec{i} + 4\,\vec{j} - 3\vec{k}$, then find $|\overrightarrow{AB}|$

Solution:

Position vector of the point A,i.e, $\overrightarrow{OA} = 2\vec{i} + \vec{j} - \vec{k}$

Position vector of the point B,i.e, $\overrightarrow{OB} = 5\,\vec{i} + 4\,\vec{j} - 3\vec{k}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= (5\,\vec{i} + 4\,\vec{j} - 3\vec{k}) - (2\vec{i} + \vec{j} - \vec{k})$$

$$= 3\,\vec{i} + 3\,\vec{j} - 2\vec{k}$$

$$|\overrightarrow{AB}| = |\overrightarrow{AB}| = \sqrt{3^2 + 3^2 + (-2)^2}$$

$$= \sqrt{9 + 9 + 4} = \sqrt{22} \text{ units}$$

PART-B

1. If A (2, 3, -4) and B (1, 0, 5) are two points, find the direction cosines and direction ratio of $\overrightarrow{AB}$

Solution:

Let $\overrightarrow{OA}$ = (2, 3, -4) = $2\vec{i}$ +$3\vec{j}$ - $4\vec{k}$

$\overrightarrow{OB}$ = (1, 0, 5) = $\vec{i}$ + 0 $\vec{j}$+$5\vec{k}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \vec{i}+5\vec{k} - (2i + 3\vec{j} - 4\vec{k})$$

$$= \vec{i} + 5k - 2\vec{i} - 3\vec{j} + 4\vec{k}$$

$$= -\vec{i} - 3\vec{j} + 9\vec{k}$$

$$r = |\overrightarrow{AB}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{(-1)^2 + (-3)^2 + 9^2}$$

$$= \sqrt{1 + 9 + 81} = \sqrt{91}$$

Direction cosines of $\overrightarrow{AB}$ are

$\cos\alpha = \frac{x}{r} = \frac{-1}{\sqrt{91}}$, $\cos\beta = \frac{y}{r} = \frac{-3}{\sqrt{91}}$ $\cos\gamma = \frac{z}{r} = \frac{9}{\sqrt{91}}$

Direction Ratio of $\overrightarrow{AB}$ are -1: -3: 9

2. If the position vector of the points A and B are $4\vec{i} - \vec{j} + 5\vec{k}$ and $3\vec{i} - 2\vec{j} - \vec{k}$ find the unit vector along $\overrightarrow{AB}$

Solution:

Let $\overrightarrow{OA} = 4\vec{i} - \vec{j} + 5\vec{k}$

$\overrightarrow{OB} = 3\vec{i} - 2\vec{j} - \vec{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= 3\vec{i} - 2\vec{j} - \vec{k} - (4\vec{i} - \vec{j} + 5\vec{k})$

$= 3\vec{i} - 2\vec{j} - \vec{k} - 4\vec{i} + \vec{j} - 5\vec{k}$

$= -\vec{i} - \vec{j} - 6\vec{k}$

r = $|\overrightarrow{AB}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{(-1)^2 + (-1)^2 + (-6)^2}$

$= \sqrt{1 + 1 + 36} = \sqrt{38}$

Unit vector $\widehat{AB} = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$

$= \frac{-\vec{i} - \vec{j} - 6\vec{k}}{\sqrt{38}}$

3. If the vectors $\vec{a} = 2\,\vec{i} - 3\,\vec{j}$ and $\vec{b} = -6\,\vec{i} + m\,\vec{j}$. are collinear, find the value of m.

Solution:

$\vec{a} = 2\,\vec{i} - 3\,\vec{j}$ and $\vec{b} = -6\,\vec{i} + m\,\vec{j}$ (since $\vec{a}$ and $\vec{b}$ are collinear)

$\vec{a} = t\vec{b}$

$2\vec{i} - 3\vec{j} = t(-6\vec{i} + m\vec{j})$

$2\vec{i} - 3\vec{j} = (-6t\vec{i} + tm\vec{j})$

Comparing coefficient of i

$2 = -6t$

$t = \frac{1}{-3}$

comparing coefficient of j

$-3 = tm$

$-3 = (\frac{1}{-3})m$

$\therefore m = 9$

PART-C

1. Show that the points whose position vectors $2i + 3j - 5\vec{k}$, $3\vec{i} + \vec{j} - 2\vec{k}$ and $6\vec{i} - 5\vec{j} + 7\vec{k}$ are collinear.

Solution: Given $\overrightarrow{OA} = 2\vec{i} + 3\vec{j} - 5\vec{k}$, $\overrightarrow{OB} = 3\vec{i} + j - 2\vec{k}$, $\overrightarrow{OC} = 6\,\vec{i} - 5\vec{j} + 7\vec{k}$

$\overrightarrow{AB} = \vec{O}B - \overrightarrow{OA}$

$= 3\vec{i} + \vec{j} - 2\vec{k} - (2\vec{i} + 3\vec{j} - 5\vec{k})$

$= 3\vec{i} + \vec{j} - 2\vec{k} - 2\vec{i} - 3\vec{j} + 5\vec{k}$

$= \vec{i} - 2\vec{j} + 3\vec{k}$ ----------> 1

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= 6\vec{i} - 5\vec{j} + 7\vec{k} - (3i + \vec{j} - 2\vec{k})$

$= 6\vec{i} - 5\vec{j} + 7\vec{k} - 3\vec{i} - \vec{j} + 2k$

$= 3\vec{i} - 6\vec{j} + 9\vec{k}$

$= 3(\vec{i} - 2\vec{j} + 3\vec{k})$

$\overrightarrow{BC} = 3\overrightarrow{AB}$ [from (1)]

$\therefore$ $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel vectors and B is the common point of these two vectors.

$\therefore$ Hence the given points A, B and C are collinear.

2. Prove that the position vectors $4\vec{i} + 5\vec{j} + 6\vec{k}$, $5\vec{i} + 6\vec{j} + 4\vec{k}$ and $6\vec{i} + 4\vec{j} + 5\vec{k}$ form the vertices of an equilateral triangle.

Solution:

Given $\overrightarrow{OA} = 4\vec{i} + 5\vec{j} + 6\vec{k}$, $\overrightarrow{OB} = 5\vec{i} + 6\vec{j} + 4\vec{k}$, $\overrightarrow{OC} = 6\vec{i} + 4\vec{j} + 5\vec{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= 5\vec{i} + 6\vec{j} + 4\vec{k} - (4\vec{i} + 5\vec{j} + 6\vec{k})$

$= 5\vec{i} + 6\vec{j} + 4\vec{k} - 4\vec{i} - 5\vec{j} - 6\vec{k}$

$= \vec{i} + \vec{j} - 2\vec{k}$

AB $= |\overrightarrow{AB}| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{1 + 1 + 4} = \sqrt{6}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= 6\vec{i} + 4\vec{j} + 5\vec{k} - (5\vec{i} + 6\vec{j} + 4\vec{k})$

$= 6\vec{i} + 4\vec{j} + 5\vec{k} - 5\vec{i} - 6\vec{j} - 4\vec{k}$

$= \vec{i} - 2\vec{j} + \vec{k}$

BC = $|\overrightarrow{BC}| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{1+4+1} = = \sqrt{6}$

$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$

$= 4\vec{i} + 5\vec{j} + 6\vec{k} - (6\vec{i} + 4\vec{j} + 5\vec{k})$

$= -2\vec{i} + \vec{j} + \vec{k}$

CA = $|\overrightarrow{CA}| = \sqrt{(-2)^2 + 1^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$

Here, AB = BC = CA = $\sqrt{6}$ (The sides are equal)

Hence, the points form an equilateral triangle.

3. Prove that the points A(2, 4, -1) , B (4, 5, 1) and C (3, 6, -3) form the vertices of a right angled isosceles triangle.

Solution:

Given:

$\overrightarrow{OA} = 2\vec{i} + 4\vec{j} - \vec{k}$, $\overrightarrow{OB} = 4\vec{i} + 5\vec{j} + \vec{k}$, $\overrightarrow{OC} = 3\vec{i} + 6\vec{j} - 3\vec{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= 4\vec{i} + 5\vec{j} + \vec{k} - (2\vec{i} + 4\vec{j} - \vec{k})$

$= 4\vec{i} + 5\vec{j} + \vec{k} - 2\vec{i} - 4\vec{j} + \vec{k}$

$= 2\vec{i} + \vec{j} + 2\vec{k}$

AB = $|\overrightarrow{AB}| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{4+1+4} = \sqrt{9} = 3$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= 3\vec{i} + 6\vec{j} - 3\vec{k} - (4\vec{i} + 5\vec{j} + \vec{k})$

$= 3\vec{i} + 6\vec{j} - 3\vec{k} - 4\vec{i} - 5\vec{j} - \vec{k}$

$= -\vec{i} + j - 4\vec{k}$

BC = $|\overrightarrow{BC}| = \sqrt{(-1)^2 + 1^2 + (-4)^2} = \sqrt{1+1+16} = \sqrt{18}$

$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$

$= 2\vec{i} + 4\vec{j} - \vec{k} - (3\vec{i} + 6\vec{j} - 3\vec{k})$

$= 2\vec{i} + 4\vec{j} - \vec{k} - 3\vec{i} - 6\vec{j} + 3\vec{k}$

$= -\vec{i} - 2\vec{j} + 2\vec{k}$

CA = $|\overrightarrow{CA}| = \sqrt{(-1)^2 + (-2)^2 + 2^2} = \sqrt{1+4+4} = \sqrt{9} = 3$

AB = CA =3

Here, $AB^2 + AC^2 = 9 + 9 = 18 = BC^2$

∴ Triangle ABC is an isosceles triangle as well as a right angled triangle with A = 90°.

4. Prove that the points A(3, -1, -2) , B (5, 1, -3) and C (6, -1, -1) form the vertices of an isosceles triangle.

Solution:

Given:

$\overrightarrow{OA} = 3\vec{i} - \vec{j} - 2\vec{k}$, $\overrightarrow{OB} = 5\vec{i} + \vec{j} - 3\vec{k}$, $\overrightarrow{OC} = 6\vec{i} - \vec{j} - \vec{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= 5\vec{i} + \vec{j} - 3\vec{k} - (3\vec{i} - \vec{j} - 2\vec{k})$

$= 5\vec{i} + \vec{j} - 3\vec{k} - 3\vec{i} + \vec{j} + 2\vec{k})$

$= 2\vec{i} + 2\vec{j} - \vec{k}$

AB = $|\overrightarrow{AB}| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{4+4+1} = \sqrt{9} = 3$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= 6\vec{i} - \vec{j} - \vec{k} - (5\vec{i} + \vec{j} - 3\vec{k})$

$= 6\vec{i} - \vec{j} - \vec{k} - 5\vec{i} - \vec{j} + 3\vec{k})$

$= \vec{i} - 2\vec{j} + 2\vec{k}$

BC = $|\overrightarrow{BC}| = \sqrt{(1)^2 + (-2)^2 + (2)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$

$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$

$= 3\vec{i} - \vec{j} - 2\vec{k} - (6\vec{i} - \vec{j} - \vec{k})$

$= 3\vec{i} - \vec{j} - 2\vec{k} - 6\vec{i} + \vec{j} + \vec{k})$

$= -3\vec{i} - \vec{k}$

CA = $|\overrightarrow{CA}|$ = $\sqrt{(-3)^2 + (-1)^2}$ = $\sqrt{9+1}$ = $\sqrt{10}$

AB = BC = $\sqrt{9}$ =3

∴ Triangle ABC is an isosceles triangle

Exercise

PART A

1. If $\vec{a} = \vec{i} + 2\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} - 4\vec{j} - 3\vec{k}$ find i) $4\vec{a} - 2\vec{b}$ ii) $5\vec{a} + 3\vec{b}$
2. If $\vec{a} = \vec{i} - 4\vec{j} + 2\vec{k}$, $\vec{b} = 3\vec{i} - 2\vec{j}$ find i) $5\vec{a} + \vec{b}$ ii) $3\vec{a} - \overrightarrow{2b}$
3. Find the unit vector along $2\vec{i}$ +$3\vec{j}$ -2 $\vec{k}$
4. Find the unit vector along $3\vec{i}$ -$\vec{j}$ -4 $\vec{k}$
5. Find the direction cosines of the vector $2\vec{i}$ -$8\vec{j}$ + $3\vec{k}$
6. Find the direction cosines of the vector $\vec{i}$ +$\vec{j}$ + $\vec{k}$
7. If the position vectors of the points A and B are $\vec{i}$ + $4\vec{j}$ – $\vec{k}$, 5 $\vec{i} - 3\,\vec{j} + 2\vec{k}$ then find $|\overrightarrow{AB}|$
8. If the position vectors of the points A and B are $2\vec{i}$ -$3\vec{j}$ +$2\vec{k}$, $\vec{i} + 4\,\vec{j} - 3\vec{k}$ then find $|\overrightarrow{AB}|$

PART B

1. If the position vectors of the points A and B are $\vec{i}$ + $\vec{j}$ + $4\vec{k}$ and $\vec{i}$ - $2\vec{j}$ + $\vec{k}$ find the direction cosines and direction ratio of $\overrightarrow{AB}$
2. If the position vectors of the points A and B are $5\vec{i}$ + $3\vec{j}$ – $\vec{k}$ and $3\vec{i}$ + $4\vec{j}$ + 6 $\vec{k}$ find the direction cosines and direction ratio of $\overrightarrow{AB}$
3. If the position vectors of the points A and B are $2\vec{i}$ + $\vec{j}$ – $\vec{k}$ and $4\vec{i}$ + $2\vec{j}$ +3 $\vec{k}$ find the direction cosines and direction ratio of $\overrightarrow{AB}$
4. If the position vectors of the points A and B are$2\vec{i}$ - $\vec{j}$ +$5\vec{k}$ and $\vec{i}$ – $\vec{j}$ + $\vec{k}$ find the unit vector
5. If the position vectors of the points A and B are$3\vec{i}$ + $\vec{j}$ -$2\vec{k}$ and $4\vec{i}$ + $\vec{j}$ -$3\vec{k}$ find the unit vector along the given vector $\overrightarrow{AB}$
6. If the position vectors of the points A and B are $\vec{i}$ - $2\vec{j}$ -$3\vec{k}$ and $4\vec{i}$ - $3\vec{j}$ + $2\vec{k}$ find the unit vector along the given vector $\overrightarrow{AB}$

PART C

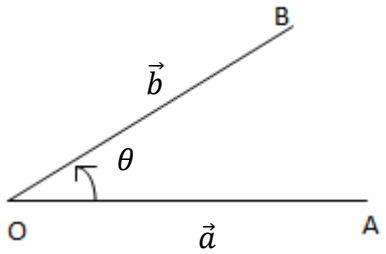
1. Prove that the position vectors $2\vec{i}$ +$3\vec{j}$ +$4\vec{k}$, $3\vec{i}$ +$4\vec{j}$ + $2\vec{k}$ and $4\vec{i}$ +$2\vec{j}$ +$3\vec{k}$ form the vertices of an equilateral triangle.
2. Prove that the points A(3, -1, -2) , B (5, 1, -3) and C (6, -1, -1) form the vertices of an isosceles triangle.
3. Prove that the points A(3, 1, -5) , B (4, 3, -7) and C (5, 2, -3) form the vertices of a right angled isosceles triangle.

5.2 SCALAR PRODUCT OF VECTORS (OR) DOT PRODUCT

If the product of two vectors $\vec{a}$ and $\vec{b}$ gives a scalar, it is called scalar product of the vectors $\vec{a}$ and $\vec{b}$ and is denoted as $\vec{a} \cdot \vec{b}$

If the angle between two vectors $\vec{a}$ and $\vec{b}$ is θ, then

$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$



Properties of scalar Product:

1. If θ is an acute angle, $\vec{a} \cdot \vec{b}$ is positive and if

   θ is an obtuse angle, $\vec{a} \cdot \vec{b}$ is negative.

2. Scalar product is commutative.

   ie) $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta = |\vec{b}|\,|\vec{a}| \cos\theta = \vec{b} \cdot \vec{a}$

   $\therefore \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

Condition for two vector to be perpendicular:

3. If $\vec{a}$ and $\vec{b}$ are (non zero) perpendicular vectors, then the angle θ between them is 90°.

   $\therefore \vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos 90^\circ = 0$ $\quad [\because \cos 90^\circ = 0]$

   If $\vec{a} \cdot \vec{b} = 0$, either $\vec{a} = 0$ or $\vec{b} = 0$ or $\vec{a}$ and $\vec{b}$ are perpendicular vector

   $\therefore$ The condition for two perpendicular vectors $\vec{a}$ and $\vec{b}$ is $\vec{a} \cdot \vec{b} = 0$

4. If $\vec{a}$ and $\vec{b}$ are parallel vectors $\theta = 0$ or 180°.

   $\therefore \vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos 0 = |\vec{a}|\,|\vec{b}|$ $\quad [\because \cos 0 = 1]$

   As a special case , $\therefore \vec{a} \cdot \vec{a} = |\vec{a}|\,|\vec{a}| = |\vec{a}|^2 = a^2$

Geometric meaning of Dot product:

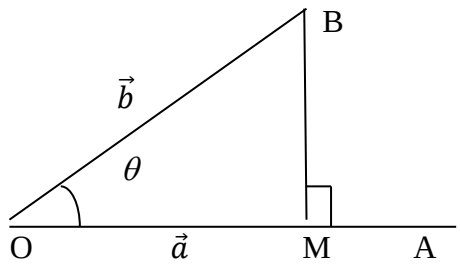
5. Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$. Draw BM perpendicular to OA.

   Let θ be the angle between $\vec{a}$ and $\vec{b}$ ie. Angle $BOA = \theta$

Now, OM is the perpendicular of $\vec{b}$ on $\vec{a}$.

From the right angled triangle BOM,

Cos $\theta = \frac{OM}{OB} = \frac{OM}{|\vec{b}|}$



$\therefore$ OM = $|\vec{b}| \cos\theta$

$= \frac{|\vec{a}|\,|\vec{b}| \cos\theta}{|\vec{a}|}$ [Multiplying Nr and Dr by $|\vec{a}|$]

$= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$ [By definition of scalar product]

$\therefore$ The projection of $\vec{b}$ on $\vec{a}$ = $\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$

Similarly, the projection of $\vec{a}$ on $\vec{b}$ = $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

6. If $\vec{i}$, $\vec{j}$, $\vec{k}$ are the unit vectors along the x, y and z axes respectively, then

$\therefore \vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1$ [using property 4]

Also $\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{i} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{j} = \vec{k} \cdot \vec{i} = \vec{i} \cdot \vec{k} = 0$ [using property 3]

Hence,

$\cdot$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	1	0	0
$\vec{j}$	0	1	0
$\vec{k}$	0	0	1

7. Dot product is distributive over addition.

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors, then $\vec{a}.(\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

8. Dot product in terms of components:

Let $\vec{a} = (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \cdot (b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$

$= a_1 b_1 (\vec{i} \cdot \vec{i}) + a_1 b_2 (\vec{i} \cdot \vec{j}) + a_1 b_3 (\vec{i} \cdot \vec{k}) + a_2 b_1 (\vec{j} \cdot \vec{i}) + a_2 b_2 (\vec{j} \cdot \vec{j}) + a_2 b_3 (\vec{j} \cdot \vec{k})$

$+ a_3 b_1 (\vec{k} \cdot \vec{i}) + a_3 b_2 (\vec{k} \cdot \vec{j}) + a_3 b_3 (\vec{k} \cdot \vec{k})$

$= a_1 b_1 + 0 + 0 + 0 + a_2 b_2 + 0 + 0 + 0 + a_3 b_3$ [By property 6]

$= a_1 b_1 + a_2 b_2 + a_3 b_3$

9. **Angle between two vectors**

We know that $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ then

$\therefore \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$

10. $(\vec{a} + \vec{b})^2 = (\vec{a} + \vec{b}).(\vec{a} + \vec{b}) = \vec{a}^2 + \vec{b}^2 + 2\vec{a}.\vec{b}$

11. $(\vec{a} - \vec{b})^2 = (\vec{a} - \vec{b}).(\vec{a} - \vec{b}) = \vec{a}^2 + \vec{b}^2 - 2\vec{a}.\vec{b}$

12. $(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = \vec{a}^2 - \vec{b}^2$

WORKED EXAMPLES
PART – A

1. Find the value of 1) $\vec{i}.\vec{i} + \vec{j}.\vec{j} + \vec{k}.\vec{k}$ 2) $\vec{i}.\vec{j} + \vec{i}.\vec{i} + \vec{j}.\vec{k}$

Solution:

1) $\vec{i}.\vec{i} + \vec{j}.\vec{j} + \vec{k}.\vec{k} = 1 + 1 + 1 = 3$
2) $\vec{i}.\vec{j} + \vec{i}.\vec{i} + \vec{j}.\vec{k} = 0 + 1 + 0 = 1$

2. Find the scalar product of the two vectors $3\vec{i} + 4\vec{j} + 5\vec{k}$ and $2\vec{i} + 3\vec{j} + \vec{k}$.

Solution:

Let $\vec{a} = 3\vec{i} + 4\vec{j} + 5\vec{k}$ and $\vec{b} = 2\vec{i} + 3\vec{j} + \vec{k}$.

$\vec{a}.\vec{b} = (3\vec{i} + 4\vec{j} + 5\vec{k}).(2\vec{i} + 3\vec{j} + \vec{k})$

$= 3(2) + 4(3) + 5(1)$

$= 6 + 12 + 5$

$= 23$

3. Find the scalar product of the two vectors $\vec{i} - \vec{j} + 3\vec{k}$ and $3\vec{i} - 7\vec{j} + \vec{k}$.

Solution:

Let $\vec{a} = \vec{i} - \vec{j} + 3\vec{k}$ and $\vec{b} = 3\vec{i} - 7\vec{j} + \vec{k}$.

$\vec{a}.\vec{b} = (\vec{i} - \vec{j} + 3\vec{k}).(3\vec{i} - 7\vec{j} + \vec{k})$

$= 1(3) + (-1)(-7) + 3(1)$

$= 3 + 7 + 3$

$= 13$

4. Prove that the vectors $3\vec{i} - \vec{j} + 5\vec{k}$ and $-6\vec{i} + 2\vec{j} + 4\vec{k}$ are perpendicular.

Solution:

Let $\vec{a} = 3\vec{i} - \vec{j} + 5\vec{k}$ and $\vec{b} = -6\vec{i} + 2\vec{j} + 4\vec{k}$

$\vec{a}.\vec{b} = (3\vec{i} - \vec{j} + 5\vec{k}).(-6\vec{i} + 2\vec{j} + 4\vec{k})$

$= 3(-6) + (-1)\,2 + 5(4)$

$= -18 - 2 + 20 = 0$

$\therefore$ the vectors $\vec{a}$ and $\vec{b}$ are perpendicular vectors.

PART-B

1. Find the projection of $2\vec{i} + \vec{j} + 2\vec{k}$ on $\vec{i} + 2\vec{j} + 2\vec{k}$

Solution:

Let $\vec{a} = 2\vec{i} + \vec{j} + 2\vec{k}$ and $\vec{b} = \vec{i} + 2\vec{j} + 2\vec{k}$

Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}\,.\,\vec{b}}{|\vec{b}|}$

$$\vec{a}\,.\,\vec{b} = \left(2\vec{i} + \vec{j} + 2\vec{k}\right).\left(\vec{i} + 2\vec{j} + 2\vec{k}\right)$$
$$= 2(1) + 1(2) + 2(2)$$
$$= 2 + 2 + 4$$
$$= 8$$
$$|\vec{b}| = \sqrt{1^2 + 2^2 + 2^2}$$
$$= \sqrt{1 + 4 + 4}$$
$$= \sqrt{9} = 3$$

Projection of $\vec{a}$ on $\vec{b} = \frac{8}{3}$

2. Find the value of ‘ p ‘ if the vectors $2\vec{i} + p\vec{j} - \vec{k}$ and $3\vec{i} + 4\vec{j} + 2\vec{k}$ are perpendicular.

Solution:

Let $\vec{a} = 2\vec{i} + p\vec{j} - \vec{k}$, $\vec{b} = 3\vec{i} + 4\vec{j} + 2\vec{k}$

Since $\vec{a}$ and $\vec{b}$ are perpendicular, $\vec{a}\,.\,\vec{b} = 0$

$(2\vec{i} + p\vec{j} - \vec{k})\,.\,(3\vec{i} + 4\vec{j} + 2\vec{k}) = 0$

$2(3) + p(4) + (-1)(2) = 0$

$6 + 4p - 2 = 0$

$4 + 4p = 0$

$4p = -4$

$P = -\frac{4}{4} = -1$

PART – C

1. Find the angle between the two vectors $\vec{i} + \vec{j} + \vec{k}$ and $3\vec{i} - \vec{j} + 2k$

Solution:

Let $\vec{a} = \vec{i} + \vec{j} + \vec{k}$, $\vec{b} = 3\vec{i} - \vec{j} + 2\vec{k}$

$\cos\theta = \frac{\vec{a}\,.\,\vec{b}}{|\vec{a}||\vec{b}|}$

$\vec{a}\,.\,\vec{b} = (\vec{i} + \vec{j} + \vec{k})\,.\,(3\vec{i} - \vec{j} + 2\vec{k})$

$= 1\,(3) + 1\,(-1) + 1\,(2)$

$= 3 - 1 + 2 = 4$

$|\vec{a}| = \sqrt{1^2 + 1^2 + 1^2}$

$= \sqrt{1^2 + 1^2 + 1^2}$

$= \sqrt{1 + 1 + 1} = \sqrt{3}$

$|\vec{b}| = \sqrt{3^2 + (-1)^2 + 2^2}$

$= \sqrt{9 + 1 + 4} = \sqrt{14}$

$\therefore \cos\theta = \frac{4}{\sqrt{3}\sqrt{14}}$

$\theta = \cos^{-1}\left(\frac{4}{\sqrt{42}}\right)$

2. Show that the vectors $2\vec{i} - 2\vec{j} + \vec{k}$, $\vec{i} + 2\vec{j} + 2\vec{k}$ and $2\vec{i} + \vec{j} - 2\vec{k}$ are mutually perpendicular (orthogonal)

Solution:

Let $\vec{a} = 2\vec{i} - 2\vec{j} + \vec{k}$, $\vec{b} = \vec{i} + 2\vec{j} + 2\vec{k}$ and $\vec{c} = 2\vec{i} + \vec{j} - 2\vec{k}$

$\vec{a}\,.\,\vec{b} = (2\vec{i} - 2\vec{j} + \vec{k})\,.\,(\vec{i} + 2\vec{j} + 2\vec{k})$

$= 2\,(1) + (-2)\,2 + 1\,(2)$

$= 2 - 4 + 2$

$= 0$

$\vec{b}\,.\,\vec{c} = (\vec{i} + 2\vec{j} + 2\vec{k})\,.\,(2\vec{i} + \vec{j} - 2\vec{k})$

$= 1\,(2) + 2\,(1) + 2(-2)$

$= 2 + 2 - 4$

$= 0$

$\vec{c}\,.\,\vec{a} = (2\vec{i} + \vec{j} - 2\vec{k})\,.\,(2\vec{i} - 2\vec{j} + \vec{k})$

$= 2\,(2) + 1\,(-2) + (-2)1$

$= 4 - 2 - 2$

$= 0$

$\vec{a}\,.\,\vec{b} = \vec{b}\,.\,\vec{c} = \vec{c}\,.\,\vec{a} = 0$

$\therefore$ Hence, the given three vectors are mutually perpendicular to each other.

Exercise

PART A

1. Find the scalar product of the two vectors $2\vec{i} - 3\vec{j} + \vec{k}$ and $3\vec{i} - 7\vec{j} + \vec{k}$.

2. Find the scalar product of the two vectors $\vec{i} - \vec{j} + 3\vec{k}$ and $\vec{i} + 3\vec{j} + 2\vec{k}$.
3. Prove that the vectors $\vec{i} + \vec{j} - 3\vec{k}$ and $3\vec{i} + 3\vec{j} + 2\vec{k}$ are perpendicular.
4. Prove that the vectors $4\vec{i} + 6\vec{j} + 3\vec{k}$ and $-3\vec{i} + \vec{j} + 2\vec{k}$ are perpendicular.
5. Prove that the vectors $4\vec{i} + \vec{j} - 3\vec{k}$ and $3\vec{i} + 3\vec{j} + 5\vec{k}$ are perpendicular.

PART-B

1. Find the value of ‘ p ‘ if the vectors $\vec{i} - 3\vec{j} + \vec{k}$ and $p\vec{i} + \vec{j} - 2\vec{k}$ are perpendicular.
2. Find the value of ‘ p ‘ if the vectors $\vec{i} + p\vec{j} + \vec{k}$ and $3\vec{i} + 4\vec{j} + 2\vec{k}$ are perpendicular.
3. Find the value of ‘ p ‘ if the vectors $2\vec{i} - 6\vec{j} - \vec{k}$ and $2\vec{i} + p\vec{j} + 3\vec{k}$ are perpendicular.
4. Find the projection of $\vec{i} + \vec{j} + 3\vec{k}$ on $-2\vec{i} + 4\vec{j} - \vec{k}$
5. Find the projection of $2\vec{i} + 3\vec{j} - \vec{k}$ on $-\vec{i} + \vec{j} - \vec{k}$
6. Find the projection of $2\vec{i} + 3\vec{j} - \vec{k}$ on $-2\vec{i} + 4\vec{j} - \vec{k}$

PART-C

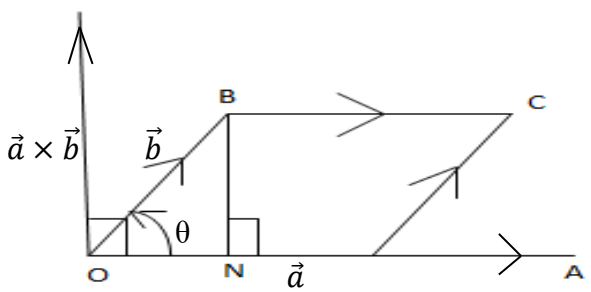
1. Find the angle between the two vectors $\vec{i} + 2\vec{j} + 8\vec{k}$ and $\vec{i} + 2\vec{j} - 2\vec{k}$
2. Find the angle between the two vectors $3\vec{i} + 4\vec{j} + 12\vec{k}$ and $-2\vec{i} + \vec{j} - \vec{k}$
3. Find the angle between the two vectors $3\vec{i} + 4\vec{j} + 12\vec{k}$ and $\vec{i} + 2\vec{j} - 2\vec{k}$
4. Show that the vectors $\vec{i} - \vec{j} + 2\vec{k}$, $4\vec{j} + 2\vec{k}$ and $7 - 10\vec{i} - 2\vec{j} + 4\vec{k}$ are mutually perpendicular (orthogonal)
5. Show that the vectors $\vec{i} + 2\vec{j} + \vec{k}$, $\vec{i} + \vec{j} - 3\vec{k}$ and $7\vec{i} - 4\vec{j} + \vec{k}$ are mutually perpendicular (orthogonal)
6. Show that the vectors $\vec{i} + 2\vec{j} + \vec{k}$, $\vec{i} + \vec{j} - 3\vec{k}$ and $7\vec{i} - 4\vec{j} + \vec{k}$ are mutually perpendicular (orthogonal)

5.3 VECTOR PRODUCT OF TWO VECTORS

Definition:

Let $\vec{a}$ and $\vec{b}$ be two vectors and θ be the angle between them. The vector product of $\vec{a}$ and $\vec{b}$ is denoted as $\vec{a} \times \vec{b}$ and it is defined as a vector $|\vec{a}|\,|\vec{b}| \sin\theta\, \hat{n}$ is the unit vector perpendicular to both $\vec{a}$, $\vec{b}$ and $\hat{n}$ are in right handed system. Thus $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\hat{n}$

The vector product is called as cross product.



Geometrical meaning of vector product:

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and θ be the angle between $\vec{a}$ and $\vec{b}$. Complete the parallelogram OACB with $\overrightarrow{OA}$ and $\overrightarrow{OB}$ as its adjacent sides.

Draw BN perpendicular to OA

From the right angled ΔONB,

$\frac{BN}{OB} = \sin\theta$

BN = OB $\sin\theta$

BN = $|\vec{b}| \sin\theta$

By definition, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta|\hat{n}|$

$= |\vec{a}|\,|\vec{b}| \sin\theta$

= OA x BN

= base x height

= Area of parallelogram OACB

$\therefore$ $|\vec{a} \text{ x } \vec{b}|$ = Area of parallelogram with $|\vec{a}|$ and $|\vec{b}|$ as adjacent sides.

Also, area of triangle OAB = $\frac{1}{2}$ (area of parallelogram OACB)

$= \frac{1}{2}|\overrightarrow{OA} \text{ x } \overrightarrow{OB}| = \frac{1}{2}|\vec{a} \text{ x } \vec{b}|$

Results :

1. The area of a parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$ is A = $|\vec{a} \text{ x } \vec{b}|$
2. The vector area of parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$ is A =$\vec{a}$ x $\vec{b}$
3. The area of a parallelogram with diagonals $\overrightarrow{d_1}$ and $\overrightarrow{d_2}$ is A = $\frac{1}{2}|\overrightarrow{d_1} \text{ x } \overrightarrow{d_2}|$
4. The area of a triangle with adjacent sides $\vec{a}$ and $\vec{b}$ is A = $\frac{1}{2}|\vec{a} \text{ x } \vec{b}|$
5. The area of triangle ABC is given by either $\frac{1}{2}|\overrightarrow{AB} \text{ x } \overrightarrow{AC}|$ (or) $\frac{1}{2}|\overrightarrow{BC} \text{ x } \overrightarrow{BA}|$ (or) $\frac{1}{2}|\overrightarrow{CA} \text{ x } \overrightarrow{CB}|$.

Properties of vector product

1. Vector product is not commutative.
   If $\vec{a}$ and $\vec{b}$ are any two vectors then, $\vec{a} \text{ x } \vec{b} \neq \vec{b} \text{ x } \vec{a}$
   however, $\vec{a} \text{ x } \vec{b}$ = - ($\vec{b} \text{ x } \vec{a}$)
2. Vector product of collinear or parallel vectors:
   If $\vec{a}$ and $\vec{b}$ are collinear or parallel, then $\theta = 0$ $\therefore \sin\theta = 0$
   $\therefore$ $|\vec{a} \text{ x } \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{n} = \vec{0}$

$\therefore \vec{a} \times \vec{b} = \vec{0}$

If $\vec{a}$ and $\vec{b}$ are collinear or parallel, then $\vec{a} \times \vec{b} = \vec{0}$

Now : If $\vec{a} \times \vec{b} = \vec{0}$ then

i) $\vec{a}$ is a zero vector and $\vec{b}$ is any vector

ii) $\vec{b}$ is a zero vector and $\vec{a}$ is any vector

iii) $\vec{a}$ and $\vec{b}$ are parallel (collinear)

3. Cross product of equal vectors

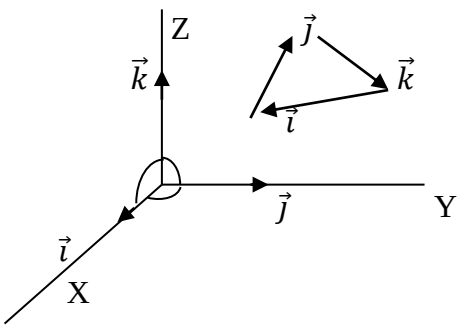
$\vec{a} \times \vec{a} = |\vec{a}|\,|\vec{a}| \sin\theta\, \hat{n}$ (here $\theta = 0°$)

$= |\vec{a}|\,|\vec{a}|\,(0)\,\hat{n}$

$= \vec{0}$

$\therefore \vec{a} \times \vec{a} = \vec{0}$ for any non zero vector $\vec{a}$

4. Cross product of unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$



Let $\vec{i}$, $\vec{j}$, $\vec{k}$ be the three mutually perpendicular unit vectors .The involvement of these unit vectors in vector product as follows : By property (3), $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$.

Also $\vec{i} \times \vec{j} = |\vec{i}|\,|\vec{j}| \sin 90^\circ\, \vec{k} = 1.1.1.\, \vec{k} = \vec{k}$ Similarly. $\vec{j} \times \vec{k} = \vec{i}$, $\vec{k} \times \vec{i} = \vec{j}$ and $\vec{j} \times \vec{i} = -\vec{k}$, $\vec{k} \times \vec{j} = -\vec{i}$, $\vec{i} \times \vec{k} = -\vec{j}$ etc. This can be shown as follows

X	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	$\vec{0}$	$\vec{k}$	$-\vec{j}$
$\vec{j}$	$-\vec{k}$	$\vec{0}$	$\vec{i}$
$\vec{k}$	$\vec{j}$	$-\vec{i}$	$\vec{0}$

5. If m is any scalar and $\vec{a}$, $\vec{b}$ are two vectors inclined at an angle θ then

$m\vec{a} \times \vec{b} = m\,(\vec{a} \times \vec{b}) = \vec{a} \times m\vec{b}$

6. **Vector product is Distributive over addition :**

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be any three vectors then

i) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$ [Left distributive]

ii) $(\vec{b} + \vec{c}) \times \vec{a} = (\vec{b} \times \vec{a}) + (\vec{c} \times \vec{a})$ [Right distributive]

7. **Vector product in determinant form :**

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \vec{i}\,(a_2 b_3 - a_3 b_2) - \vec{j}\,(a_1 b_3 - a_3 b_1) + \vec{k}(a_1 b_2 - a_2 b_1)$$

8. **Sine of Angle between two vectors:**

Let $\vec{a}$, $\vec{b}$ be two vectors inclined at an angle θ, then

$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\ \hat{n}$

$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$, $\therefore \sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$

$\theta = \sin^{-1}\left(\frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}\right)$

9. **Unit vector perpendicular to two vectors**

Let $\vec{a}$ and $\vec{b}$ be two non zero, and non parallel vectors and θ be the angle between them

$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\ \hat{n}$ ----------> (1)

$\therefore |\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$ ----------> (2)

$\therefore$ (1) $\div$ (2) gives $\frac{|\vec{a}|\,|\vec{b}| \sin\theta\ \hat{n}}{|\vec{a}|\,|\vec{b}| \sin\theta} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

$\Rightarrow \hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

Note that , $-\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$ is also a unit vector perpendicular to $\vec{a}$ and $\vec{b}$

$\therefore$ Unit vector perpendicular to $\vec{a}$ and $\vec{b}$ are $\pm\ \hat{n} = \pm\ \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

WORKED EXAMPLES

PART – A

1. Prove that $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$

Solution:

$$\begin{aligned} \text{LHS} &= (\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) \\ &= (\vec{a} \times \vec{a}) + (\vec{a} \times \vec{b}) - (\vec{b} \times \vec{a}) - (\vec{b} \times \vec{b}) \\ &= (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{b}) \\ &= 2(\vec{a} \times \vec{b}) = \text{RHS} \end{aligned}$$

2. Prove that $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = 0$

Solution:

$$\begin{aligned} \text{LHS} &= \vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) \\ &= (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c}) + (\vec{b} \times \vec{a}) + (\vec{c} \times \vec{a}) + (\vec{c} \times b) \\ &= (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c}) - (\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c}) - (\vec{b} \times \vec{c}) \\ &= 0 = \text{RHS} \end{aligned}$$

3. If $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$, $\vec{b} = \vec{i} - 2\vec{j} + 3\vec{k}$, then find $\vec{a} \times \vec{b}$

Solution:

Given that $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$, $\vec{b} = \vec{i} - 2\vec{j} + 3\vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} -1 & 1 \\ -2 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix}$$

$$= \vec{i}\,(-3 + 2) - \vec{j}\,(6 - 1) + \vec{k}\,(-4 + 1)$$

$$= \vec{i}\,(-1) - \vec{j}\,(5) + \vec{k}\,(-3)$$

$$= -\vec{i} - 5\vec{j} - 3\vec{k}$$

4. If $|\vec{a}| = 2$, $|\vec{b}| = 7$ and $|\vec{a} \times \vec{b}| = 7$, then find the angle between $\vec{a}$ and $\vec{b}$

Solution:

Given that $|\vec{a}| = 2$, $|\vec{b}| = 7$ and $|\vec{a} \times \vec{b}| = 7$

$$\sin\vartheta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$$

$$= \frac{7}{2 \times 7} = \frac{1}{2}$$

$$\theta = \sin^{-1}\left(\frac{1}{2}\right) = 30^\circ$$

PART-B

1. Find the area of the parallelogram whose adjacent sides are $3\vec{i} - \vec{j} - 2\vec{k}$ and $5\vec{i} + 6\vec{j} - 7\vec{k}$

Solution:

Let $\vec{a} = 3\vec{i} - \vec{j} - 2\vec{k}$ and $\vec{b} = 5\vec{i} + 6\vec{j} - 7\vec{k}$

Area of parallelogram A = $|\vec{a} \times \vec{b}|$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & -2 \\ 5 & 6 & -7 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} -1 & -2 \\ 6 & -7 \end{vmatrix} - \vec{j}\begin{vmatrix} 3 & -2 \\ 5 & -7 \end{vmatrix} + \vec{k}\begin{vmatrix} 3 & -1 \\ 5 & 6 \end{vmatrix}$$

$$= \vec{i}\,(7 + 12) - \vec{j}\,(-21 + 10) + \vec{k}\,(18 + 5)$$

$$= \vec{i}\,(19) - \vec{j}\,(-11) + \vec{k}\,(23)$$

$$= 19\vec{i} + 11\vec{j} + 23\vec{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{19^2 + 11^2 + 23^2}$$

$$= \sqrt{361 + 121 + 529} = \sqrt{1011} \text{ sq. uints}$$

2. Find the area of the parallelogram whose diagonals are $\vec{i} + 3\vec{j} + \vec{k}$ and $3\vec{i} - \vec{j} + 2\vec{k}$

Solution:

Let $\overrightarrow{d_1} = \vec{i} + 3\vec{j} + \vec{k}$ and $\overrightarrow{d_2} = 3\vec{i} - \vec{j} + 2\vec{k}$

Area of parallelogram A = $\frac{1}{2}|\overrightarrow{d_1} \times \overrightarrow{d_2}|$

$$\overrightarrow{d_1} \times \overrightarrow{d_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 1 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix}$$

$$= \vec{i}\,(6 + 1) - \vec{j}\,(2 - 3) + \vec{k}\,(-1 - 9)$$

$$= \vec{i}\,(7) - \vec{j}\,(-1) + \vec{k}\,(-10)$$

$$= 7\vec{i} + \vec{j} - 10\vec{k}$$

$$\left|\overrightarrow{d_1} \times \overrightarrow{d_2}\right| = \sqrt{7^2 + 1^2 + (-10)^2}$$

$$= \sqrt{49 + 1 + 100} = \sqrt{150}$$

$$A = \frac{1}{2}\left|\overrightarrow{d_1} \times \overrightarrow{d_2}\right|$$

$$= \frac{1}{2}\sqrt{150} \text{ sq. uints}$$

3. Find the area of the triangle whose adjacent sides are $\vec{i} - 2\vec{j} + 4\vec{k}$ and $4\vec{i} - \vec{j} + 5\vec{k}$

Solution:

Let $\vec{a} = \vec{i} - 2\vec{j} + 4\vec{k}$ and $\vec{b} = 4\vec{i} - \vec{j} + 5\vec{k}$

Area of Triangle $A = \frac{1}{2}\left|\vec{a} \times \vec{b}\right|$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 4 \\ 4 & -1 & 5 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} -2 & 4 \\ -1 & 5 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 4 \\ 4 & 5 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & -2 \\ 4 & -1 \end{vmatrix}$$

$$= \vec{i}\,(-10 + 4) - \vec{j}\,(5 - 16) + \vec{k}\,(-1 + 8)$$

$$= \vec{i}\,(-6) - \vec{j}\,(-11) + \vec{k}\,(7)$$

$$= -6\vec{i} + 11\vec{j} + 7\vec{k}$$

$$\left|\vec{a} \times \vec{b}\right| = \sqrt{(-6)^2 + 11^2 + 7^2}$$

$$= \sqrt{36 + 121 + 49} = \sqrt{206}$$

$$A = \frac{1}{2}\left|\vec{a} \times \vec{b}\right|$$

$$= \frac{1}{2}\sqrt{206} \text{ sq. uints}$$

PART – C

1. Find the unit vector perpendicular to each of the vectors $\vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{i} - \vec{j} - \vec{k}$. Also find the sine of the angle between these two vectors.

Solution:

Let $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$, $\vec{b} = \vec{i} - \vec{j} - \vec{k}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}$$

$= \vec{i}\,(-2 + 3) - \vec{j}\,(-1 - 3) + \vec{k}\,(-1 - 2)$

$= \vec{i}\,(1) - \vec{j}\,(-4) + \vec{k}\,(-3) = \vec{i} + 4\vec{j} - 3\vec{k}$

$|\vec{a} \times \vec{b}| = \sqrt{x^2 + y^2 + z^2}$

$= \sqrt{1^2 + (4)^2 + (-3)^2}$

$= \sqrt{1 + 16 + 9}$

$= \sqrt{26}$

$|\vec{a}| = \sqrt{1^2 + 2^2 + 3^2}$

$= \sqrt{1 + 4 + 9}$

$= \sqrt{14}$

$|\vec{b}| = \sqrt{1^2 + (-1)^2 + (-1)^2}$

$= \sqrt{1 + 1 + 1}$

$= \sqrt{3}$

The unit vector perpendicular to $\vec{a}$ and $\vec{b}$ is given by $\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

$\hat{n} = \frac{\vec{i} + 4\vec{j} - 3\vec{k}}{\sqrt{26}}$

The sine of angle between $\vec{a}$ and $\vec{b}$ is given by $\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$

$\therefore \sin\theta = \frac{\sqrt{26}}{\sqrt{14}\,\sqrt{3}}$

$\theta = sin^{-1}\left(\frac{\sqrt{26}}{\sqrt{14}\,\sqrt{3}}\right)$

2. Find the unit vector perpendicular to each of the vectors $2\vec{i} - \vec{j} + \vec{k}$ and $3\vec{i} + 4\vec{j} - \vec{k}$. Also find the sine of the angle between these two vectors.

Solution:

Let $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$ and $\vec{b} = 3\vec{i} + 4\vec{j} - \vec{k}$.

$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix}$

$= \vec{i}\begin{vmatrix} -1 & 1 \\ 4 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix}$

$= \vec{i}\,(1 - 4) - \vec{j}\,(-2 - 3) + \vec{k}\,(8 + 3)$

$= \vec{i}\,(-3) - \vec{j}\,(-5) + \vec{k}\,(11) = -3\vec{i} + 5\vec{j} + 11\vec{k}$

$|\vec{a} \times \vec{b}| = \sqrt{x^2 + y^2 + z^2}$

$= \sqrt{(-3)^2 + (5)^2 + (11)^2}$

$= \sqrt{9 + 25 + 121}$

$= \sqrt{155}$

$|\vec{a}| = \sqrt{2^2 + (-1)^2 + 1^2}$

$= \sqrt{4 + 1 + 1}$

$= \sqrt{6}$

$|\vec{b}| = \sqrt{3^2 + (4)^2 + (-1)^2}$

$= \sqrt{9 + 16 + 1}$

$= \sqrt{26}$

The unit vector perpendicular to $\vec{a}$ and $\vec{b}$ is given by $\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

$\hat{n} = \frac{-3\vec{i} + 5\vec{j} + 11\vec{k}}{\sqrt{155}}$

The sine of angle between $\vec{a}$ and $\vec{b}$ is given by $\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$

$\therefore \sin\theta = \frac{\sqrt{155}}{\sqrt{6}\,\sqrt{26}}$

$\theta = sin^{-1}\left(\frac{\sqrt{155}}{\sqrt{6}\,\sqrt{26}}\right)$

3. If the points of the triangle whose position vectors are $2\vec{i} + 3\vec{j} + 4\vec{k}$, $3\vec{i} + 4\vec{j} + 2\vec{k}$ *and* $4\vec{i} + 2\vec{j} + 3\vec{k}$, then find the area of the triangle

Solution:

Let $\overrightarrow{OA} = 2\vec{i} + 3\vec{j} + 4\vec{k}$, $\overrightarrow{OB} = 3\vec{i} + 4\vec{j} + 2\vec{k}$, $\overrightarrow{OC} = 4\vec{i} + 2\vec{j} + 3\vec{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= 3\vec{i} + 4\vec{j} + 2\vec{k} - (2\vec{i} + 3\vec{j} + 4\vec{k})$

$= 3\vec{i} + 4\vec{j} + 2\vec{k} - 2\vec{i} - 3\vec{j} - 4\vec{k}$

$\overrightarrow{AB} = \vec{i} + \vec{j} - 2\vec{k}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= 4\vec{i} + 2\vec{j} + 3\vec{k} - (3\vec{i} + 4\vec{j} + 2\vec{k})$

$= 4\vec{i} + 2\vec{j} + 3\vec{k} - 3\vec{i} - 4\vec{j} - 2\vec{k}$

$\overrightarrow{BC} = \vec{i} - 2\vec{j} + \vec{k}$

Area of Triangle A = $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}$$

$= \vec{i}\,(1 - 4) - \vec{j}\,(1 + 2) + \vec{k}\,(-2 - 1)$

$= \vec{i}\,(-3) - \vec{j}\,(3) + \vec{k}\,(-3) = -3\vec{i} - 3\vec{j} - 3\vec{k}$

$\left|\overrightarrow{AB} \times \overrightarrow{BC}\right| = \sqrt{x^2 + y^2 + z^2}$

$= \sqrt{(-3)^2 + (-3)^2 + (-3)^2}$

$= \sqrt{9 + 9 + 9}$

$= \sqrt{27}$

Area of Triangle A = $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$

$= \frac{1}{2}\sqrt{27}$ sq. uints

Exercise

Part A

1. If $|\vec{a}| = 2$, $|\vec{b}| = 7$, and $|\vec{a} \text{ X } \vec{b}| = 7$, then find the angle between $\vec{a}$ and $\vec{b}$
2. If $|\vec{a}| = 4$, $|\vec{b}| = 14$, and $|\vec{a} \text{ X } \vec{b}| = 14$, then find the angle between $\vec{a}$ and $\vec{b}$
3. If $|\vec{a}| = 3$, $|\vec{b}| = 7$ and $|\vec{a} \text{ X } \vec{b}| = 14$ find the angle between $\vec{a}$ and $\vec{b}$
4. Prove that $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = 2(\vec{b} \times \vec{a})$
5. If $\vec{a} = 2\vec{i} + 5\vec{j} - \vec{k}$, $\vec{b} = -2\vec{i} - 5\vec{k}$, find $\vec{a}$ x $\vec{b}$

Part B

1. Find the area of the parallelogram whose adjacent sides are $\vec{i} - 2\vec{j} - 3\vec{k}$, $\vec{i} + 6\vec{j} + 4\vec{k}$
2. Find the area of the parallelogram whose adjacent sides are $3\vec{i} + 4\vec{j} - \vec{k}$, $\vec{i} + \vec{j} - \vec{k}$
3. Find the area of the parallelogram whose diagonals are $2\vec{i} + 3\vec{j} + \vec{k}$, $\vec{i} - 4\vec{j} + 2\vec{k}$
4. Find the area of the parallelogram whose diagonals are $3\vec{i} + 4\vec{j} + \vec{k}$, $3\vec{i} - \vec{j} + 2\vec{k}$
5. Find the area of the triangle whose adjacent sides are $\vec{i} - 2\vec{j} + 4\vec{k}$, $4\vec{i} - \vec{j} + 5\vec{k}$
6. Find the area of the triangle whose adjacent sides are $2\vec{i} - \vec{j} + \vec{k}$, $\vec{i} + 3\vec{j} + \vec{k}$

Part C

1. If the points of the triangle whose position vectors are $4\vec{i} + 2\vec{j} + \vec{k}$, $-2\vec{i} - 2\vec{j} + \vec{k}$ *and* $5\vec{i} + 2\vec{j} + 3\vec{k}$, then find the area of the triangle
2. If the points of the triangle whose position vectors are $2\vec{i} + 7\vec{j} + 4\vec{k}$, $\vec{i} - \vec{j} + \vec{k}$ *and* $2\vec{i} + 4\vec{j} + \vec{k}$, then find the area of the triangle
3. Find the unit vector perpendicular to each of the vectors $-\vec{i} + 2\vec{j} + 5\vec{k}$ and $3\vec{i} - \vec{j} + 2\vec{k}$.Also find the sine of the angle between these two vectors.
4. Find the unit vector perpendicular to each of the vectors $\vec{i} + \vec{j} + \vec{k}$ and $2\vec{i} - \vec{j} + 3\vec{k}$.Also find the sine of the angle between these two vectors.
5. Find the unit vector perpendicular to each of the vectors $3\vec{i} + \vec{j} + 2\vec{k}$ and $2\vec{i} - 2\vec{j} + 4\vec{k}$.Also find the sine of the angle between these two vectors.
6. If the points of the triangle whose position vector $\vec{i} + 3\vec{j} + 4\vec{k}$, $2\vec{i} - \vec{j} + \vec{k}$ *and* $-\vec{i} + 2\vec{j} + 3\vec{k}$ find the area of the triangl

MODEL QUESTION - I

ENGINEERING MATHEMATICS

Time - Three Hours
(Maximum marks: 75)

[N.B:- (1) Answer any FIVE Questions in each of PART-A & PART –B and any two Divisions of each question in PART-C

(2) Each question carries 2(two) marks in PART –A, 3(three) marks in PART –B and 5(five) marks for each division in PART- C]

Part – A

1. Solve : $\begin{vmatrix} x & 5 \\ 20 & 2 \end{vmatrix} = 0.$

2. Solve : $x^3 - 1 = 0$.

3. Prove that $\sin 72° \cos 18° + \cos 78° \sin 18° = 1$

4. Evaluate: $\lim_{\theta \to 2} \frac{\sin 2\theta}{\theta}$.

5. If $y = 2x^5 - cosecx$, find $\frac{dy}{dx}$

6. If u= $x^3 - 5x^2y + y^3$, find $\frac{\partial u}{\partial y}$

7. Evaluate: $\int tan^4 x \, sec^2 x \, dx$

8. Prove that the vectors $2\vec{i} - 5\vec{j} + \vec{k}$ and $3\vec{i} + \vec{j} - \vec{k}$ are perpendicular.

Part – B

9. Find the rank of the matrix $\begin{pmatrix} 6 & 12 & 6 \\ 7 & 2 & 1 \\ 4 & 8 & 4 \end{pmatrix}$.

10. Find the modulus and amplitude of $\sqrt{3} + i$.

11. If $a = \cos\alpha + isin\,\alpha, b = \cos\beta + isin\,\beta$,then prove that $\frac{a}{b} + \frac{b}{a} = 2cos(\alpha - \beta)$

12. If tan A= $\frac{1}{2}$ and tan B = $\frac{1}{3}$,find the value of tan (A+B) .

13. Find $\frac{dy}{dx}$,if y = $x^3 \log x sin\, x$

14. Evaluate: $\int \sqrt{1 + \sin 2x}\, dx$

15. Find the modulus and direction cosines of the vector $5\vec{i} - 2\vec{j} + 3\vec{k}$

16. Find the area of the triangle two of whose sides are the vectors $3\vec{i} - 2\vec{j} - \vec{k}$ and $\vec{i} - 2\vec{j} + \vec{k}$

[Turn over ……

Part – C

17. a) Solve the following equations by using Cramer's rule

$$3x + y - z = 2\ ;\ \ 2x - y + 2z = 6\ ;\ 2x + y - 2z = -2$$

b) Find the Inverse of the matrix $\begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{pmatrix}$

c) Find the real and imaginary parts of the complex number $\frac{(1+i)(2-i)}{1-i}$

18. a) simplify: $\frac{(\cos 3\theta - i\sin 3\theta)^5(\cos 4\theta + i\sin 4\theta)^4}{(\cos 2\theta + i\sin 2\theta)^{-7}(\cos 3\theta - i\sin 3\theta)^6}$

b) Solve: $x^7 + 1 = 0$

c) Prove that $(\cos A - \cos B)^2 + (\sin A - \sin B)^2 = 4\sin^2\left(\frac{A-B}{2}\right)$

19. a) Evaluate : $\lim_{x\to 2} \frac{x^3-2^3}{x^4-2^4}$

b) Find $\frac{dy}{dx}$ for the following :

i) $y = \log(\sin x)$ ii) $y = \frac{1-\sin x}{1+\sin x}$

c) If $y = x^2 \cos x$,then prove that $x^2 y_2 - 4xy_1 + (x^2+6)y = 0$

20 a) If $u = \log(x^2+y^2)$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2$

b) Evaluate the following :

i) $\int (3x^2 - 5sec^2 x + \frac{7}{x})dx$ ii) $\int \frac{sin^2 x}{1+\cos x}\, dx$

c) Evaluate the following :

i) $\int \frac{6x+4}{3x^2+4x+5} dx$ ii) $\int \frac{\sin(\log x)}{x} dx$

21 a) Prove that the position vectors $3\vec{i} + \vec{j} + 2\vec{k},\ \vec{i} + 2\vec{j} + 3\vec{k},\ and\ 2\vec{i} + 3\vec{j} + \vec{k}$ form the vertices of

an equilateral triangle.

b) Find the angle between the vectors $3\vec{i} + 4\vec{j} + 12\vec{k}$ and $\vec{i} + 2\vec{j} + 2\vec{k}$

c) Find the unit vector perpendicular to each of the vectors $\vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{i} - \vec{j} - \vec{k}$.

Also find the sine of the angle between them.

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MODEL QUESTION - II

ENGINEERING MATHEMATICS

Time - Three Hours
(Maximum marks: 75)

[N.B:- (1) Answer any FIVE Questions in each of PART-A & PART –B and any two Divisions of each question in PART-C

(2) Each question carries 2(two) marks in PART –A, 3(three) marks in PART –B and 5(five) marks for each division in PART- C]

Part – A

1. Solve : $\begin{vmatrix} x & 9 \\ 4 & x \end{vmatrix} = 0.$

2. Simplify: $\omega^4 + \omega^5 + \omega^6$

3. Find the value of cos20° cos 10° - sin 20° sin 10°.

4. Evaluate: $\lim_{x \to 3} \frac{x^2 - 3^2}{x-3}$.

5. Find $\frac{dy}{dx}$, if y = sin 2x

6. If u= $x^2 + 3xy + y^2$, find $\frac{\partial u}{\partial x}$

7. Evaluate: $\int (x^2 - 3x + 2)(2x - 3)\, dx$

8. If $\vec{a} = 5\vec{i} - \vec{j} - 6\vec{k}$ and $\vec{b} = -7\vec{i} + 3\vec{j} - 2\vec{k}$, find $\vec{a} \cdot \vec{b}$

Part – B

9. Find the inverse of the matrix $\begin{pmatrix} 4 & 5 \\ -6 & 3 \end{pmatrix}$

10. Find the real and imaginary parts of the complex number $\frac{5+i}{2+3i}$

11. If $a = \cos\alpha + i\sin\alpha, b = \cos\beta + i\sin\beta$, find the value of $ab - \frac{1}{ab}$

12. If tan A= $\frac{5}{6}$ and tan B = $\frac{1}{11}$, then Prove that A+B = 45°.

13. Find $\frac{dy}{dx}$, if y = $\frac{\sin x}{e^x}$

14. Evaluate: $\int (\tan x + \cot x)^2 dx$

15. The position vector of the points A and B are $3\vec{i} - \vec{j} + 2\vec{k}$ and $2\vec{i} + \vec{j} + \vec{k}$, find $|\overrightarrow{AB}|$

16. Find the area of the parallelogram whose adjacent sides are $\vec{i} + \vec{j} + \vec{k}$ and $3\vec{i} + 2\vec{j} - \vec{k}$

[Turn over ……

Part – C

17. a) Solve the following equations by using Cramer's rule

$$x + y + z = 3\ ;\quad 2x - y + z = 2\ ;\quad 3x + 2y - 2z = 3$$

b) Find the rank of the matrix $\begin{pmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{pmatrix}$

c) Find the modulus and amplitude of $\frac{1+i}{1-i}$

18. a) simplify: $\frac{(\cos 2\theta - i\sin 2\theta)^4(\cos 4\theta + i\sin 4\theta)^{-5}}{(\cos 3\theta + i\sin 3\theta)^2(\cos 5\theta - i\sin 5\theta)^{-3}}$ when $\theta = \frac{\pi}{49}$

b) Solve: $x^6 - 1 = 0$

c) Prove that $(\cos A + \cos B)^2 + (\sin A + \sin B)^2 = 4\cos^2\left(\frac{A-B}{2}\right)$

19. a) Evaluate : $\lim_{\theta \to 0} \frac{\sin 6\theta}{\sin 7\theta}$

b) Find $\frac{dy}{dx}$ for the following :

i) $y = 3x^2 + e^x - \sin x$ ii) $y = x \log x \cos x$

c) If $y = x^2 \sin x$, then prove that $x^2 y_2 - 4xy_1 + (x^2 + 6)y = 0$

20 a) If $u = x^2 + 4xy + y^2$, then prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

b) Evaluate the following :

i) $\int (x^2 + 2x + 3)dx$ ii) $\int \cos^2 x\, dx$

c) Evaluate the following :

i) $\int \frac{3x^2 - 6}{x^3 - 6x + 4} dx$ ii) $\int \sin^3 x \cos x\, dx$

21 a) Prove that the position vectors $2\vec{i} + 10\vec{k},\ -4\vec{i} + 5\vec{j} + 6\vec{k},\ and\ 7\vec{i} + 6\vec{j} + 6\vec{k}$ form the vertices of an isosceles triangle.

b) Prove that the vectors $\vec{i} + 2\vec{j} + \vec{k},\ \vec{i} + \vec{j} - 3\vec{k},\ and\ 7\vec{i} - 4\vec{j} + \vec{k}$ are mutually perpendicular.

c) Find the unit vector perpendicular to each of the vectors $3\vec{i} + \vec{j} + 2\vec{k}$ and $2\vec{i} - 2\vec{j} + 4\vec{k}$.

Also find the sine of the angle between them.

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